Intuition for Disaster Measure

Suppose that you are interested in an *ex ante* measure of underlying return variability such as the variance of holding period return[1]

$$\mathbb{V} \equiv Var_0\left(\ln\left(\frac{S_T}{S_0}\right)\right)$$

By definition, for a pure diffusion, a VIX type calculation is an estimate of integrated variance (see details on the next page)

$$\mathbb{IV} \equiv E_0\left[\int_0^T \sigma_t^2 dt\right]$$

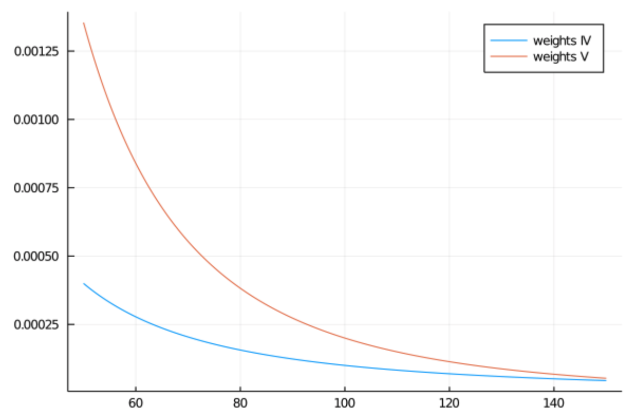
Moreover, for a pure diffusion the two quantities are exactly equal $\mathbb{V} = \mathbb{IV}$.

For a process with jumps, however, $\mathbb{V}$ consists of two components

$$Var_0\left(\ln\left(\frac{S_T}{S_0}\right)\right) = \int_0^T \sigma_t^2 dt + \text{Jump Component}$$

As a result, a VIX type calculation is a biased estimate of both the $\mathbb{IV}$ (the first component) and a biased estimate of $\mathbb{V}$ (the total).

The solution proposed by Du and Kapadia (2012) is to directly use a property of variance ($V(X) = E[X^2] - E[X]^2$) where each moment can be replicated using the *option spanning theorem*. Such calculation results in a different weighting scheme for options at different strikes. In particular, it puts a relatively higher weight on OTM puts. I show these weights as a function of strike K on the figure below for spot price $S_0 = 100$ (note that $\mathbb{IV}$ weights are the standard $1/K^2$)



Of course, we can always put more weight on OTM puts just because we want to capture higher order moments. Importantly, however, Du and Kapadia (2012) show that the difference between the two measures is related only to the jump component of the process so that there is a nice theoretical motivation for changing the weighting scheme in this particular way.

$$\mathbb{V} - \mathbb{IV} = 2\underbrace{\left(\int_R \psi(x) f(x) dx\right)}_{\text{Jump magnitude}}\underbrace{\left(E_0\int_0^T \lambda_t dt\right)}_{\text{Jump Intensity}}$$

where $\psi(x) = 1 + x + \frac{1}{2}x^2 - e^x$, $f(x)$ is the density of the size of the jump and λ_t is intensity of jumps. Thus, the difference consists of two components

1. A component related to the to the size of the jump. For negative jumps ($x < 0$) $\psi(x)$ increases as x gets more negative thus putting a relatively larger weight on larger jumps.

2. A the component related to expected jump intensity going forward.

[1] here and going forward everything is under a risk neutral measure

Additional Details

Pure diffusion Consider a pure diffusion process

$$\frac{dS_t}{S_t} = rdt + \sigma_t dZ_t \Leftrightarrow d\ln(S_t) = \left(r - \frac{1}{2}\sigma_t^2\right) + \sigma_t dZ_t$$

and take the difference to obtain

$$\sigma_t^2 = 2\left(\frac{dS_t}{S_t} - d\ln(S_t)\right)$$

Integrated variance (IV) is

$$IV \equiv E_0\left[\frac{1}{T}\int_0^T \sigma_t^2 dt\right] = E_0\left[\frac{2}{T}\int_0^T \left(\frac{dS_t}{S_t} - d\ln(S_t)\right)\right] = \frac{2}{T}\left(rT - E_0\left[\ln\frac{S_T}{S_0}\right]\right).$$

The latter expectation can be estimated using options and *option spanning theorem*

$$E_0\left[\ln\frac{S_T}{S_0}\right] = \cdots - \int_0^{S^*} \frac{1}{K^2}P(K)dK - \int_{S^*}^{\infty} \frac{1}{K^2}C(K)dK$$

and this is precisely how VIX is calculated. Importantly, when the process doesn't feature any jumps IV is equal to the variance of holding period return

$$IV = Var_0\left(\ln\left(\frac{S_T}{S_0}\right)\right)$$

Hence, using a VIX type estimate will give a correct variance when there are no jumps.

Jump-diffusion In the presence of jumps, the variance of holding period return

$$Var_0\left(\ln\left(\frac{S_T}{S_0}\right)\right) = \int_0^T \sigma_t^2 dt + \text{Jump Component}$$

The general formula for variance is

$$V \equiv Var_0\left(\ln\left(\frac{S_T}{S_0}\right)\right) = E_0\left[\ln\left(\frac{S_T}{S_0}\right)^2\right] - E_0\left[\ln\left(\frac{S_T}{S_0}\right)\right]^2$$

where the first term features a squared log contract that can be also calculated using options

$$E_0\left[\ln\left(\frac{S_T}{S_0}\right)^2\right] = ... + \int_0^{S^*} \frac{2(1-\ln K/S_0)}{K^2}P(K)dK + \int_{S^*}^{\infty} \frac{2(1-\ln K/S_0)}{K^2}C(K)dK$$

where one can notice that the weighting scheme of options of different strikes differs from the VIX type estimation.