Online Appendix to Portfolio Choice with Sustainable Spending: A Model of Reaching for Yield

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This online appendix contains notes on data collection, proofs of propositions from the main text as well as additional details on derivations omitted in main text. The order and section numbering follow the main text.

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1 Data Sources

To construct a 20-year constant maturity TIPS yield, we use TIPS yields from FRED. First, we linearly interpolate TIPS yields of the nearest maturities for each date available and then aggregate to the monthly level by taking an average. We construct the final yearly series shown in the upper panel of Figure 1 by taking the June observation for each year.

We obtain data on the asset allocation of endowments and sovereign wealth funds from their annual reports. We use the actual asset allocation from the highlights section of the Yale Endowment Annual Report (link), the target asset allocation from the "Report from Stanford Management Company" section of the Stanford Treasurer Annual Reports (link), the actual asset allocation of the General Endowment Pool of the University of California System (link), the actual asset allocation from the University of Kansas System annual reports (link), the actual asset allocation from annual reports of the Alaska Permanent Fund (link), the actual asset allocation from annual reports of the Singapore GIC (link), the actual asset allocation from annual reports of the Australian Future Fund (link), and the target asset allocation from annual reports of the Norwegian Oil Fund (link). For the value-weighted asset allocation of US endowments we use public tables from NACUBO's study of endowments (link). We construct the risky portfolio share as 100% less allocations to fixed-income securities and cash.

Expected Returns and Risks of Portfolios Figure 1 from the main text shows how the risky share of endowments and sovereign funds evolved over time. Here we use a more detailed information on their portfolios and assumptions on expected returns and covariances for different asset classes to calculate how expected returns, risk and Sharpe Ratios of these portfolios.

Perfold and Stafford (2010) obtain 2004 forecasts from internal documents of Harvard Management Company (HMC) that manages Harvard University's endowment. In particular, their Exhibit 17 contains expected returns, standard deviations and correlations of returns for 13 asset classes: *domestic equity, foreign equity, emerging markets equity, private equity, absolute return, high yield, commodities, natural resources, real estate, domestic bonds, foreign bonds, inflation indexed bonds and cash.* First, we use expected return on cash as the risk free return and calculate excess return for each asset class by subtracting the expected return for cash. Second, we match institutions' asset classes to HMC's asset classes. While in most cases it is straightforward, not all institutions report their portfolio weights with the same level of details as HMC. For example, an institution might report overall allocation to bonds without splitting into domestic and foreign bonds. In such cases, we assign HMC's domestic bonds asset class to overall bond portfolio share and we do the same for equity. The results are not sensitive to the exact matching of asset classes. Finally, we use portfolio weights to calculate time series of excess return, standard deviation and Sharpe ratio of their portfolios using HMC's forecasts.

Figure A.1 presents the time series of excess returns, standard deviations and Sharpe ratios for each institution described above. The excess return series in the top panel has the same trends as the risky share series from Figure 1 in the main text. The differences between excess returns and standard deviation presented in the middle panel are easiest to see through Sharpe ratio presented in the bottom panel. For example, NACUBO series features a large increase in Sharpe ratio in the first half of the sample and a subsequent decline. There has been a large increase in Sharpe Ratio for University of California system endowment (UC). At the same time, Target allocation for Stanford features a declining Sharpe ratio in the later part of the sample.

Are Institutions' Allocations Consistent with Merton Rule? Standard Merton rule prescribes a risky share $\alpha = \frac{\mu}{\gamma\sigma^2}$. Rearranging we get

$$\underbrace{\alpha\sigma}_{\text{Portfolio Risk}} = \frac{1}{\gamma} \underbrace{\frac{\mu}{\sigma}}_{\text{Portfolio Sharpe Ratio}}.$$

Taking logs and differences we see that the change in log risk of the portfolio should be equal to the change in log Sharpe ratio if the investor follows Merton rule

$$\Delta \log(\text{Portfolio Risk}) = \Delta \log(\text{Portfolio SR}). \tag{A.1}$$

Using data on institutions' portfolios we can calculate left and right sides of equation (A.1) where the change is calculated by taking the difference between the last and first observation for each portfolio. We compare the two quantities in Figure A.2. For NACUBO that represents a dollar-weighted asset allocation of US endowments changes in log risk and log Sharpe ratio are very close to one another. However, for the majority of institutions including Stanford, Yale, Alaska Permanent Fund, Norway Petroleum Fund and Australian Future Fund, the increase in portfolio risk significantly exceeded the increase in portfolio Sharpe Ratio. This violates the Merton rule and points to excessive risk taking possibly occurring due to reaching for yield.

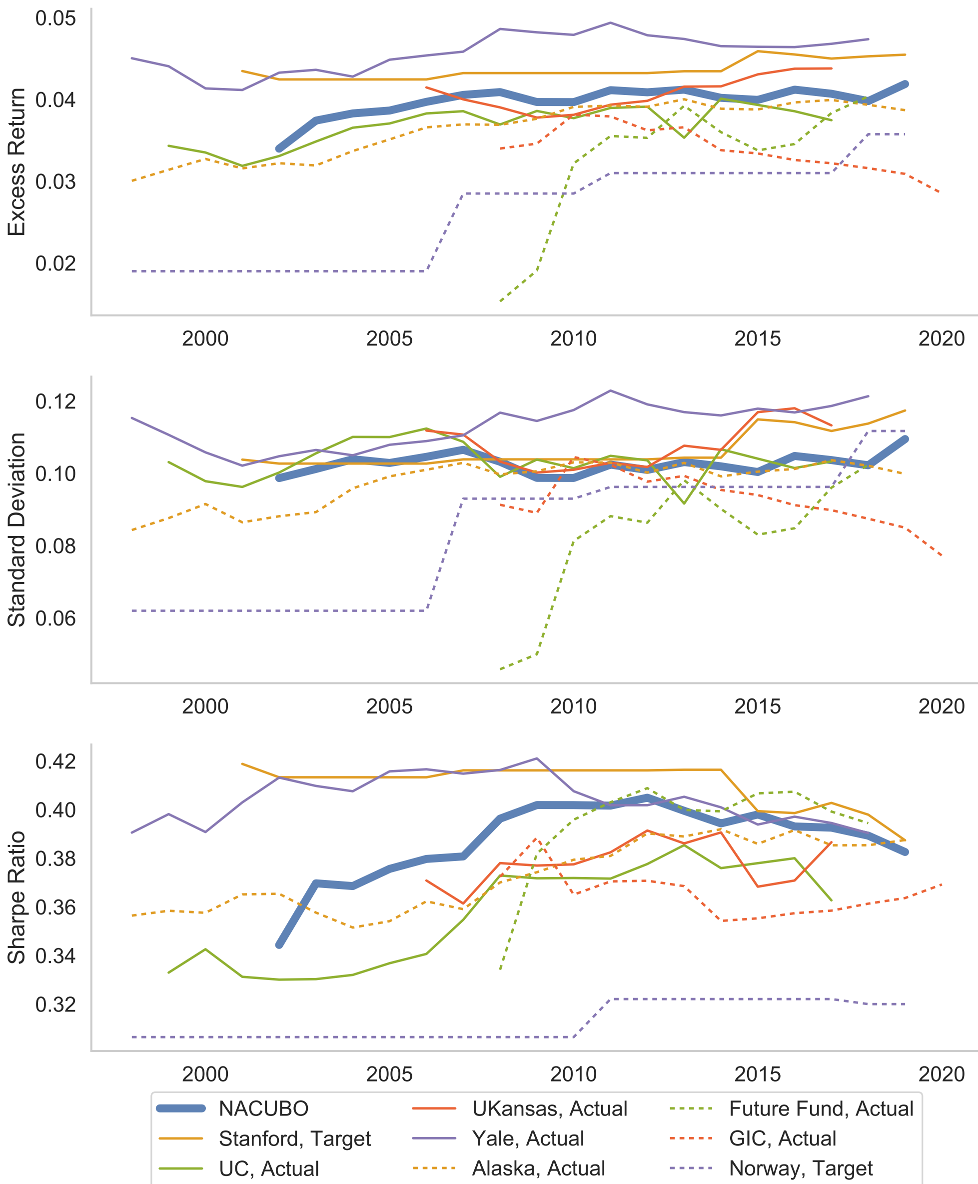

Figure A.1: Endowments and Sovereign Wealth Funds Portfolio Characteristics

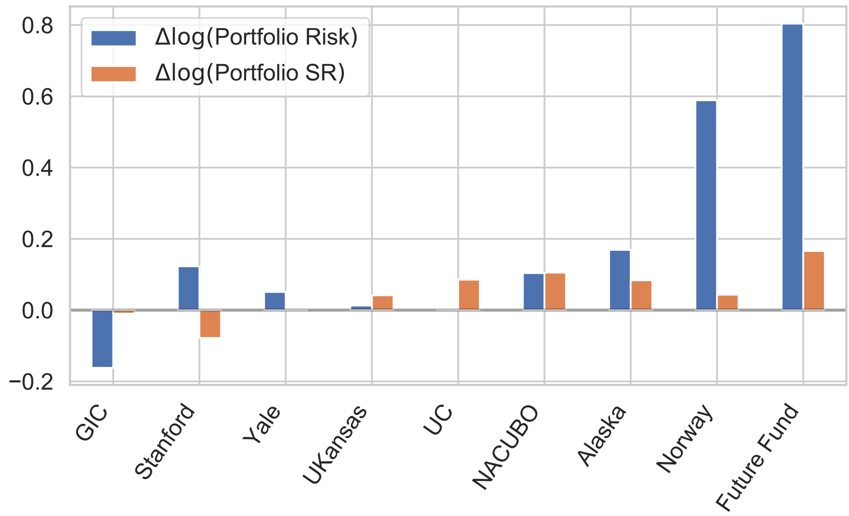

Figure A.2: Change in Risk and Sharpe Ratio of Institutions' Portfolios

2 Comparative Statics with Power Utility

2.1 The Standard Unconstrained Model

Here we show how to derive a closed-form solution for the agent's lifetime utility for given values of the consumption-wealth ratio θ and the risky share α. Given a process for consumption

$$\frac{dc_t}{c_t} = (r_f + \alpha\mu)dt + \alpha\sigma dZ_t - \theta dt,$$

we can write the process for log consumption as

$$d\log c_t = \left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)dt + \alpha\sigma dZ_t - \theta dt.$$

Iterating this expression forward we get

$$\log c_t = \log c_0 + \int_0^t d\log c_s = \log c_0 + \left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2 - \theta\right)t + \alpha\sigma Z_t.$$

The expectation $\mathrm{E}_0 c_t^{1-\gamma}$ is

$$\begin{aligned}
\mathrm{E}_0 c_t^{1-\gamma} &= \mathrm{E}_0 e^{(1-\gamma)\log c_t} \\
&= e^{(1-\gamma)E_0[\log c_t]+\frac{1}{2}(1-\gamma)^2 Var_0(\log c_t)} \\
&= e^{(1-\gamma)\log c_0+(1-\gamma)\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2-\theta\right)t+\frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2 t} \\
&= c_0^{1-\gamma} e^{(1-\gamma)\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2-\theta\right)t+\frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2 t}
\end{aligned}$$

Lifetime utility is then

$$\begin{aligned}
v &= E_0 \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \\
&= \frac{1}{1-\gamma} \int_0^\infty e^{-\rho t} E_0 \left[c_t^{1-\gamma}\right] dt \\
&= \frac{c_0^{1-\gamma}}{1-\gamma} \int_0^\infty e^{-\rho t} e^{(1-\gamma)\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2-\theta\right)t+\frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2 t} dt \\
&= \frac{c_0^{1-\gamma}}{1-\gamma} \int_0^\infty e^{-\left(\rho-(1-\gamma)\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2-\theta\right)-\frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2\right)t} dt \\
&= \frac{c_0^{1-\gamma}}{1-\gamma} \left(- \frac{e^{-\left(\rho-(1-\gamma)\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2-\theta\right)-\frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2\right)t}}{\rho-(1-\gamma)\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2-\theta\right)-\frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2} \right)\Bigg|_{t=0}^{\infty} \\
&= \frac{w_0^{1-\gamma}}{1-\gamma} \frac{\theta^{1-\gamma}}{\rho-(1-\gamma)\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2-\theta\right)-\frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2}.
\end{aligned} \tag{A.2}$$

The first-order condition for α is

$$-(1-\gamma)(\mu-\alpha\sigma^2)-(1-\gamma)^2\alpha\sigma^2=0,$$

$$-\mu+\alpha\sigma^2-\alpha\sigma^2+\gamma\alpha\sigma^2=0 \Rightarrow \alpha=\frac{\mu}{\gamma\sigma^2}.$$

The first-order condition for θ is

$$\frac{(1-\gamma)\theta^{-\gamma}\left(\rho-(1-\gamma)\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2-\theta\right)-\frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2\right)-(1-\gamma)\theta^{1-\gamma}}{(denominator)^2}=0.$$

Substituting in the optimal portfolio rule $\alpha=\mu/\gamma\sigma^2$ and cancelling $(1-\gamma)\theta^{-\gamma}$:

$$\left(\rho-(1-\gamma)\left(r_f+\frac{\mu^2}{\gamma\sigma^2}-\frac{1}{2}\frac{\mu^2}{\gamma^2\sigma^2}-\theta\right)-\frac{1}{2}(1-\gamma)^2\frac{\mu^2}{\gamma^2\sigma^2}\right)-\theta=0,$$

$$\theta = \frac{\rho}{\gamma} + \frac{\gamma - 1}{\gamma}\left(r + \frac{1}{2\gamma}\left(\frac{\mu}{\sigma}\right)^2\right).$$

Substituting α and θ into the law of motion for consumption (and wealth) we get

$$\begin{aligned}\frac{dc_t}{c_t} = \frac{dw_t}{w_t} &= \left(r_f + \frac{\mu^2}{\gamma\sigma^2} - \frac{\rho}{\gamma} - \frac{\gamma - 1}{\gamma}\left(r + \frac{1}{2\gamma}\left(\frac{\mu}{\sigma}\right)^2\right)\right) + \frac{\mu}{\gamma\sigma} dZ_t \\ &= \left(\frac{r_f - \rho}{\gamma} + \frac{1 + \gamma}{2\gamma^2}\left(\frac{\mu}{\sigma}\right)^2\right) dt + \frac{1}{\gamma}\left(\frac{\mu}{\sigma}\right) dZ_t\end{aligned}$$

2.2 An Arithmetic Sustainable Spending Constraint: Proof of Proposition 1

To prove proposition 1 we derive a closed form expression for the risky share α. The first-order condition of the problem with the arithmetic constraint,

$$\max_{\alpha} \ v = \max_{\alpha} \ \frac{w_0^{1-\gamma}}{1-\gamma} \frac{(r_f + \alpha\mu)^{1-\gamma}}{\rho - \frac{1}{2}\gamma(\gamma - 1)\alpha^2\sigma^2},$$

is

$$\frac{(1-\gamma)(r_f + \alpha\mu)^{-\gamma}\mu\left(\rho - \frac{1}{2}\gamma(\gamma-1)\alpha^2\sigma^2\right) + (r_f + \alpha\mu)^{1-\gamma}\gamma(\gamma-1)\alpha\sigma^2}{(denominator)^2} = 0.$$

We cancel $(1-\gamma)(r_f + \alpha\mu)^{-\gamma}$ to obtain the following quadratic equation:

$$\mu\left(\rho - \frac{1}{2}\gamma(\gamma-1)\alpha^2\sigma^2\right) - (r_f + \alpha\mu)\gamma\alpha\sigma^2 = 0,$$

$$-\alpha^2 \cdot \frac{1}{2}\gamma(\gamma+1)\mu\sigma^2 - \alpha \cdot r_f\gamma\sigma^2 + \mu\rho = 0.$$

Two solutions for this quadratic equation are

$$\alpha = \frac{-r_f\gamma\sigma^2 \pm \sqrt{(r_f\gamma\sigma^2)^2 + 4\frac{1}{2}\gamma(\gamma+1)\mu^2\sigma^2\rho}}{\gamma(\gamma+1)\mu\sigma^2},$$

$$\alpha = \frac{-r_f \pm \sqrt{(r_f)^2 + 2\rho\frac{(\gamma+1)}{\gamma}\left(\frac{\mu}{\sigma}\right)^2\rho}}{(\gamma+1)\mu}.$$

We are interested in the solution where $\alpha > 0$ so that the second-order condition is satisfied. Therefore, we take the largest solution with the positive sign:

$$\alpha = \frac{-r_f + \sqrt{(r_f)^2 + 2\rho\frac{(\gamma+1)}{\gamma}\left(\frac{\mu}{\sigma}\right)^2}}{(\gamma+1)\mu}.$$

Effect of the rate of time preference We see that K is increasing in ρ so that α will be increasing in ρ :

$$\frac{d\alpha}{d\rho} > 0.$$

This means that a more impatient investor has a more aggressive asset allocation.

Effect of the riskfree rate We can also see the effect of the riskfree rate on asset allocation using the solution above. This is equivalent to considering the derivative of

$$-r_f + \sqrt{r_f^2 + X}$$

with respect to r_f where X is a constant. Let's take that derivative and compare it:

$$-1 + \frac{r_f}{\sqrt{r_f^2 + X}} \; vs. \; 0,$$

$$\frac{r_f}{\sqrt{r_f^2 + X}} \; vs. \; 1.$$

We can see that the left-hand side is smaller than one meaning that the derivative of α w.r.t. r_f is negative.

Taking the second derivative we can see that the relationship between α and r_f is convex:

$$\frac{\sqrt{r_f^2 + X} - \frac{r_f^2}{\sqrt{r_f^2+X}}}{r_f^2 + X} = \frac{(r_f^2 + X) - r_f^2}{(r_f^2 + X)^{3/2}} > 0$$

Effect of the risk premium Now we consider the effect of the risk premium on the risky share. We use the first-order condition and implicit function theorem to write

$$f(\alpha, \mu) \equiv \alpha^2 \cdot \mu(1+\gamma)\gamma\sigma^2 + \alpha \cdot 2r_f\gamma\sigma^2 - 2\rho\mu = 0.$$

$$\frac{d\alpha}{d\mu} = -\frac{\partial f/\partial \mu}{\partial f/\partial \alpha} = -\frac{\alpha^2(1+\gamma)\gamma\sigma^2 - 2\rho}{2\alpha\mu(1+\gamma)\gamma\sigma^2 + 2r\gamma\sigma^2}.$$

The first-order condition allows us to sign the numerator:

$$\mu(\alpha^2 \cdot \mu(1+\gamma)\gamma\sigma^2 - 2\rho) = -\alpha \cdot 2r_f\gamma\sigma^2$$

Under $\mu > 0$ we have $\alpha > 0$ and, therefore, the numerator is positively proportional to $-r_f$. We use the notation $\propto$ to denote this positive proportionality. Now we work with the denominator

$$\begin{aligned}\frac{\partial f}{\partial \alpha} &= 2\alpha\mu(1+\gamma)\gamma\sigma^2 + 2r_f\gamma\sigma^2 \\ &\propto \alpha\mu(1+\gamma) + r_f \\ \text{[Use solution for } \alpha] &= \frac{-r_f + \sqrt{K}}{\mu(1+\gamma)}\mu(1+\gamma) + r_f \\ &= \sqrt{K} > 0\end{aligned}$$

Combining both results we get our comparative static:

$$\frac{d\alpha}{d\mu} \propto r_f,$$

where as already noted we use $\propto$ to denote positive proportionality.

2.3 A Geometric Sustainable Spending Constraint: Proof of Proposition 2

The first-order condition for the problem with a geometric constraint,

$$\max_{\alpha} v = \max_{\alpha} \frac{w_0^{1-\gamma}}{1-\gamma} \frac{\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{1-\gamma}}{\rho - \frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2}, \tag{A.3}$$

is

$$\frac{(1-\gamma)\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{-\gamma}(\mu - \alpha\sigma^2)\left(\rho - \frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2\right) + (1-\gamma)^2\alpha\sigma^2\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{1-\gamma}}{(denominator)^2} = 0.$$

We cancel $(1-\gamma)\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{-\gamma}$ to get

$$(\mu - \alpha\sigma^2)\left(\rho - \frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2\right) + (1-\gamma)\alpha\sigma^2\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right) = 0.$$

Note that for $\gamma = 1$ we recover the growth optimal asset allocation $\alpha = \mu/\sigma^2$. We next consider the case when $\gamma > 1$ and return to $\gamma < 1$ later.

For certain parts of the derivations it will be easier to work with a modified version of the first-order condition. We divide through by $(1-\gamma)^2\alpha\sigma^2$, rearrange and define

$$h \equiv \frac{\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)}{1-\gamma} - \frac{1}{2}\alpha(\mu - \alpha\sigma^2) + \rho\frac{(\mu - \alpha\sigma^2)}{(1-\gamma)^2\alpha\sigma^2}. \tag{A.4}$$

The first order condition is then $h = 0$.

We now characterize α using the implicit function theorem. It says that

$$\frac{d\alpha}{d\beta} = -\frac{\partial h/\partial\beta}{\partial h/\partial\alpha},$$

where β is a variable of interest like the riskfree rate.

Since all comparative statics depend on $\partial h/\partial\alpha$ we sign this first.

$$\begin{aligned}
\frac{\partial h}{\partial\alpha} &= \frac{\mu}{1-\gamma} - \frac{\alpha\sigma^2}{1-\gamma} - \frac{1}{2}(\mu - \alpha\sigma^2) - \frac{1}{2}\alpha(-\sigma^2) - \frac{\rho\mu}{(1-\gamma)^2\alpha^2\sigma^2} \\
&= \frac{1}{1-\gamma}(\mu - \alpha\sigma^2) - \frac{1}{2}(\mu - \alpha\sigma^2) + \frac{1}{2}\alpha\sigma^2 - \frac{\rho\mu}{(1-\gamma)^2\alpha^2\sigma^2} \\
&= \left(\frac{1}{1-\gamma} - \frac{1}{2}\right)(\mu - \alpha\sigma^2) + \frac{1}{2}\alpha\sigma^2 - \frac{\rho\mu}{(1-\gamma)^2\alpha^2\sigma^2} \\
&= \frac{1+\gamma}{2(1-\gamma)}(\mu - \alpha\sigma^2) + \frac{1}{2}\alpha\sigma^2 - \frac{\rho\mu}{(1-\gamma)^2\alpha^2\sigma^2}
\end{aligned}$$

To proceed, we go back to equation (A.4) and note that

$$\underbrace{\frac{\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)}{1-\gamma}}_{<0 \text{ for } \gamma>1} - \frac{1}{2}\alpha(\mu - \alpha\sigma^2) + \rho\frac{(\mu - \alpha\sigma^2)}{(1-\gamma)^2\alpha\sigma^2} = 0.$$

Therefore, we should have

$$-\frac{1}{2}\alpha(\mu - \alpha\sigma^2) + \rho\frac{(\mu - \alpha\sigma^2)}{(1-\gamma)^2\alpha\sigma^2} > 0,$$

$$(\mu - \alpha\sigma^2)\left(\frac{\rho}{(1-\gamma)^2\alpha\sigma^2} - \frac{1}{2}\alpha\right) > 0,$$

$$\underbrace{(\mu - \alpha\sigma^2)}_{>0 \text{ for } \gamma>1 \text{ from eq. (1)}} \underbrace{\frac{1}{(1-\gamma)\alpha\sigma^2}}_{<0 \text{ for } \gamma>1}\left(\rho\frac{1}{1-\gamma} - \frac{1}{2}(1-\gamma)\alpha^2\sigma^2\right) > 0,$$

$$\implies \rho\frac{1}{1-\gamma} - \frac{1}{2}(1-\gamma)\alpha^2\sigma^2 < 0 \implies \frac{1}{2} < \frac{\rho}{(1-\gamma)^2\alpha^2\sigma^2}.$$

We multiply both sides by $-\mu$ to finally get

$$-\frac{\rho}{(1-\gamma)^2\alpha^2\sigma^2} < -\frac{1}{2}\mu$$

and use this for our comparative static

$$\begin{aligned}
\frac{\partial h}{\partial \alpha} &< \frac{1+\gamma}{2(1-\gamma)}(\mu - \alpha\sigma^2) + \frac{1}{2}\alpha\sigma^2 - \frac{1}{2}\mu \\
&= \frac{1+\gamma}{2(1-\gamma)}(\mu - \alpha\sigma^2) - \frac{1}{2}(\mu - \alpha\sigma^2) \\
&= \left(\frac{1+\gamma}{2(1-\gamma)} - \frac{1}{2}\right)(\mu - \alpha\sigma^2) \\
&= \underbrace{\frac{\gamma}{1-\gamma}}_{<0}\underbrace{(\mu - \alpha\sigma^2)}_{>0} < 0 \\
\implies \frac{\partial h}{\partial \alpha} &< 0
\end{aligned} \tag{A.5}$$

Thus we have evaluated the denominator of the comparative static and can simplify it to

$$\frac{d\alpha}{d\beta} \propto \frac{\partial h}{\partial \beta}.$$

Riskfree rate From equation (A.4) $\partial h/\partial r_f < 0$, implying that

$$\frac{d\alpha}{dr_f} < 0.$$

The risky share α decreases in the riskfree rate r_f.

Convexity in the riskfree rate We next prove that $\alpha(r_f)$ is a convex function. To do this we differentiate h from equation (A.4) w.r.t. r_f twice to get

$$0 = h(\alpha(r_f), r_f)$$

$$0 = \left(\frac{\partial^2 h}{\partial \alpha^2}\frac{d\alpha}{dr_f} + \frac{\partial^2 h}{\partial r_f \partial \alpha}\right)\frac{d\alpha}{dr_f} + \frac{\partial h}{\partial \alpha}\frac{d^2\alpha}{dr_f^2} + \frac{\partial^2 h}{\partial r_f \partial \alpha}\frac{d\alpha}{r_f} + \frac{\partial^2 h}{\partial r_f^2}$$

From equation (A.4) we know that $\frac{\partial h}{\partial r_f} = \frac{1}{1-\gamma}$, therefore, $\frac{\partial^2 h}{\partial r_f \partial \alpha} = \frac{\partial^2 h}{\partial \alpha \partial r_f} = \frac{\partial^2 h}{\partial r_f^2} = 0$. We get

$$0 = \frac{\partial^2 h}{\partial \alpha^2}\left(\frac{d\alpha}{dr_f}\right)^2 + \frac{\partial h}{\partial \alpha}\frac{d^2\alpha}{dr_f^2},$$

$$\frac{d^2\alpha}{dr_f^2} = -\frac{\frac{\partial^2 h}{\partial \alpha^2}\left(\frac{d\alpha}{dr_f}\right)^2}{\frac{\partial h}{\partial \alpha}}.$$

We already signed $\frac{\partial h}{\partial \alpha} < 0$. Hence,

$$\frac{d^2\alpha}{dr_f^2} \propto \frac{\partial^2 h}{\partial \alpha^2} = -\frac{1+\gamma}{2(1-\gamma)}\sigma^2 + \frac{\sigma^2}{2} + 2\frac{\rho\mu}{(1-\gamma)^2\alpha^3\sigma^2} > 0,$$

since $\gamma > 1$ and $\alpha > 0$.

Rate of time preference From equation (A.4) $\partial h/\partial \rho < 0$, implying that

$$\frac{d\alpha}{d\rho} > 0.$$

The risky share α increases in the discount rate ρ.

Risk premium First we find a value of the riskfree rate r_f^* such that the risky share does not depend on the risk premium. Using equation (A.4) once again and collecting the terms with μ,

$$h(\alpha, \mu) = \frac{r_f}{1-\gamma} + \mu \left(\frac{\alpha}{1-\gamma} - \frac{\alpha}{2} + \frac{\rho}{(1-\gamma)^2 \alpha \sigma^2} \right) - \frac{1}{2} \frac{\alpha^2 \sigma^2}{1-\gamma} + \frac{1}{2} \alpha^2 \sigma^2 - \frac{\rho}{(1-\gamma)^2} = 0.$$

If the optimal α does not depend on μ, then the expression multiplying μ should equal zero. This gives us a condition for the risky share,

$$\frac{\alpha}{1-\gamma} - \frac{\alpha}{2} + \frac{\rho}{(1-\gamma)^2 \alpha \sigma^2} = 0,$$

$$\alpha^2 \frac{1+\gamma}{2(1-\gamma)} + \frac{\rho}{(1-\gamma)^2 \sigma^2} = 0,$$

$$\alpha = \alpha^* \equiv \sqrt{\frac{2\rho}{(\gamma^2 - 1)\sigma^2}},$$

where we pick a positive solution.

Substituting this into the first-order condition, we can derive the expression for the riskfree rate that makes α indifferent to μ:

$$\frac{r_f}{1-\gamma} - \frac{1}{2} \frac{\alpha^2 \sigma^2}{1-\gamma} + \frac{1}{2} \alpha^2 \sigma^2 - \frac{\rho}{(1-\gamma)^2} = 0,$$

$$r_f - \frac{1}{2} \alpha^2 \sigma^2 + \frac{1}{2} \alpha^2 \sigma^2 (1-\gamma) - \frac{\rho}{1-\gamma} = 0,$$

$$r_f - \gamma \frac{1}{2} \alpha^2 \sigma^2 + \frac{\rho}{\gamma - 1} = 0,$$

$$r_f - \gamma \frac{\rho}{(\gamma^2 - 1)} + \frac{\rho}{\gamma - 1} = 0,$$

$$r_f = \frac{\gamma \rho}{(\gamma^2 - 1)} - \frac{\rho}{\gamma - 1} = 0,$$

$$r_f = \frac{\gamma \rho - \rho(\gamma + 1),}{(\gamma^2 - 1)}$$

$$r_f = r_f^* \equiv -\frac{\rho}{\gamma^2 - 1} < 0.$$

Note that for $\gamma > 1$, $r_f^* < 0$.

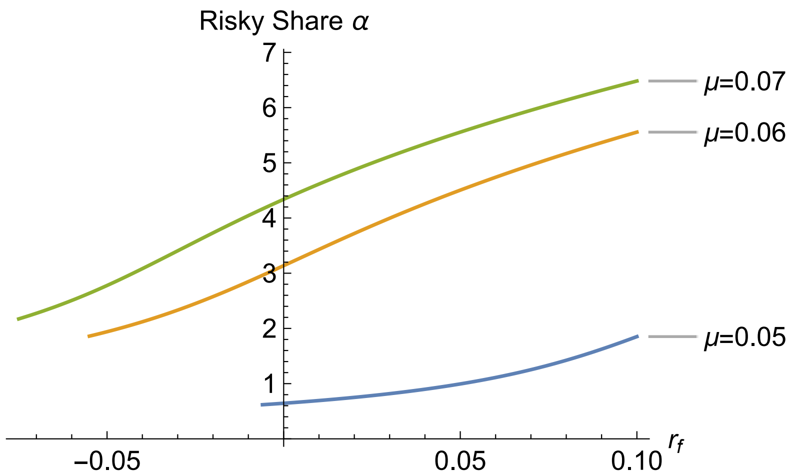

Figure A.3: Optimal Risky Share α for $\gamma < 1$

We verify that the comparison of r_f with r_f^* determines whether α increases or decreases with μ. Using the implicit function theorem we have

$$\frac{d\alpha}{d\mu} = -\frac{\partial h/\partial \mu}{\partial h/\partial \alpha}.$$

From previous derivations we know that $\partial h/\partial \alpha < 0$. Therefore

$$\begin{aligned} \frac{d\alpha}{d\mu} \propto \frac{\partial h}{\partial \mu} &= \frac{\alpha}{1-\gamma} - \frac{\alpha}{2} + \frac{\rho}{(1-\gamma)^2 \alpha \sigma^2} \\ &= \alpha \frac{1+\gamma}{2(1-\gamma)} + \frac{\rho}{(1-\gamma)^2 \alpha \sigma^2} \\ &\propto -\alpha^2 \frac{1+\gamma}{2(\gamma-1)} + \frac{\rho}{(\gamma-1)^2 \sigma^2} = \begin{cases} > 0 \text{ for } \alpha > \alpha^* \\ < 0 \text{ for } \alpha < \alpha^* \end{cases} \end{aligned}$$

where α^* is defined above. Since α decreases in r_f, $r_f > r_f^*$ implies that $\alpha < \alpha^*$. In this region $\frac{\partial h}{\partial \mu} > 0 \implies \frac{d\alpha}{d\mu} > 0$: the optimal risky share increases in the risk premium. When $\alpha > \alpha^*$, which happens when $r_f < r_f^*$, we have that $\frac{\partial h}{\partial \mu} < 0 \implies \frac{d\alpha}{d\mu} < 0$: the optimal risky share decreases in the risk premium.

Proofs for $\gamma < 1$. Figure 1 from the main text presents the optimal risky share as a function of the risk free rate r_f for different values of risk premium μ. Figure A.3 presents an analogous figure for the case when $\gamma < 1$.

First consider equation (A.4). When $\gamma < 1$, α and $\mu - \alpha\sigma^2$ should be of opposite signs. Therefore, when $\alpha > 0$ (a sufficient second order condition) we have $\mu - \alpha\sigma^2 < 0$. Using equation (A.4) we can sign

$$\underbrace{\frac{\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)}{1-\gamma}}_{>0} - \frac{1}{2}\alpha(\mu - \alpha\sigma^2) + \rho\frac{(\mu - \alpha\sigma^2)}{(1-\gamma)^2\alpha\sigma^2} = 0$$

$$\Rightarrow \left(-\frac{1}{2}\alpha + \rho\frac{1}{(1-\gamma)^2\alpha\sigma^2}\right)\underbrace{(\mu - \alpha\sigma^2)}_{<0} < 0 \Rightarrow \rho > \frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2$$

which is exactly the same condition that we derived for $\gamma > 1$. Therefore, we can proceed with signing $\frac{\partial h}{\partial \alpha}$ in a similar way as we did for the case when $\gamma > 1$ to obtain

$$\frac{\partial h}{\partial \alpha} < \underbrace{\frac{\gamma}{1-\gamma}}_{>0}\underbrace{(\mu - \alpha\sigma^2)}_{<0} < 0$$

so that a comparative static w.r.t. any parameter β is $\frac{d\alpha}{d\beta} \propto \frac{\partial h}{\partial \beta}$. We get

$$\frac{d\alpha}{dr_f} \propto \frac{\partial h}{\partial r_f} = \frac{1}{1-\gamma} > 0,$$

$$\frac{d\alpha}{d\rho} \propto \frac{\partial h}{\partial \rho} = \frac{\mu - \alpha\sigma^2}{(1-\gamma)^2\alpha\sigma^2} < 0,$$

$$\frac{d\alpha}{d\mu} \propto \frac{\partial h}{\partial \mu} = \frac{\alpha}{1-\gamma} - \frac{1}{2}\alpha + \frac{\rho}{(1-\gamma)^2\alpha\sigma^2} = \underbrace{\frac{1+\gamma}{2(1-\gamma)}}_{>0}\alpha + \underbrace{\frac{\rho}{(1-\gamma)^2\alpha\sigma^2}}_{>0} > 0.$$

We see that the effects of r_f and ρ are reversed, so that a lower riskfree rate and a higher rate of time preference lead to a lower risky share. The effect of the risk premium is now positive for all levels of the riskfree rate.

2.4 The Welfare Cost of Sustainable Spending

We define lifetime value as a function of arbitrary risky share α, consumption wealth ratio θ, risk free rate r_f and initial wealth w_0 as in equation (A.2):

$$v(\alpha, \theta, w_0) \equiv \frac{w_0^{1-\gamma}}{1-\gamma}\frac{\theta^{1-\gamma}}{\rho - (1-\gamma)\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2 - \theta\right) - \frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2}.$$

In line with the main text, the welfare loss from the sustainable spending constraint is λ that solves

$$v(\alpha^{UC}, \theta^{UC}, (1-\lambda)w_0) = v(\alpha^C, \theta^C, w_0)$$

where UC's denote the unconstrained (Merton) parameters and C's denote constrained parameters. We can then explicitly solve for λ as a function of $(\alpha^{UC}, \theta^{UC}, \alpha^C, \theta^C)$. For arithmetic constraint we have closed form expressions for α^C and θ^C, for geometric constraint we calculate α^C and θ^C numerically.

Welfare loss with Merton portfolio rule If the agent in addition is constrained to have a Merton portfolio choice rule $\alpha = \alpha^{UC} = \frac{\mu}{\gamma\sigma^2}$, his consumption to wealth ratio is $\theta(\alpha^{UC}) = r_f + \alpha^{UC}\mu$ under arithmetic constraint and $\theta(\alpha^{UC}) = r_f + \alpha^{UC}\mu - \frac{1}{2}(\alpha^{UC})^2\sigma^2$ under geometric constraint. The welfare loss is defined as λ that solves

$$v(\alpha^{UC}, \theta^{UC}, (1-\lambda)w_0) = v(\alpha^{UC}, \theta(\alpha^{UC}), w_0)$$

As previously, we can solve for λ in closed form

3 Extensions of the Static Model

3.1 A One-Sided Sustainable Spending Constraint

We derive the level of the interest rate that makes the constraint non-binding in the sense that the constrained agent behaves as if he is unconstrained and has the same portfolio allocation and consumption-wealth ratio.

Arithmetic average model Consider an arithmetic average model where we have a closed-form solution. Equating the risky share for the arithmetic model and the Merton portfolio rule we get a condition

$$\frac{-r_f + \sqrt{r_f^2 + 2\rho\frac{1+\gamma}{\gamma}\left(\frac{\mu}{\sigma}\right)^2}}{\mu(1+\gamma)} = \frac{\mu}{\gamma\sigma^2},$$

that simplifies to

$$r_f = \rho - \frac{1+\gamma}{2\gamma}\left(\frac{\mu}{\sigma}\right)^2. \tag{A.6}$$

Geometric average model For the geometric average model, consider equation (A.4) that implicitly defines the risky share α and substitute $\alpha = \mu/\gamma\sigma^2$ to get a condition

$$\left(r_f + \frac{\mu}{\gamma\sigma^2}\mu - \frac{1}{2\gamma^2}\frac{\mu^2}{\sigma^2}\right) - \rho\frac{\mu - \mu/\gamma}{(\gamma-1)\mu/\gamma} - \frac{1}{2\gamma}\frac{\mu}{\sigma^2}\left(\mu - \frac{\mu}{\gamma}\right)(1-\gamma) = 0,$$

that simplifies to

$$r_f = \rho - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 \tag{A.7}$$

As we discuss in the main text, under our baseline parameter assumptions this level of the riskfree rate is close to 2%.

3.2 Donations

Arithmetic average model In the presence of donations, the budget constraint and the arithmetic consumption rule become

$$dw_t = w_t dr_{p,t} + w_t(g_u + g_e) - c_t dt$$

$$c_t dt = w_t(E_t dr_{p,t} + g_u) = w_t(r_f + g_u + \alpha\mu)dt$$

We substitute the consumption rule into the budget constraint to obtain

$$\begin{aligned} dw_t &= w_t(r_f + \alpha\mu) + w_t\alpha\sigma dZ_t + w_t(g_u + g_e) - w_t(r_f + g_u + \alpha\mu)dt \\ &= w_t g_e dt + w_t\alpha\sigma dZ_t \end{aligned}$$

The process for log consumption coincides with the process for log wealth

$$d\log(w_t) = d\log(c_t) = \left(g_e - \frac{1}{2}\alpha^2\sigma^2\right) dt + \alpha\sigma dZ_t$$

such that the portfolio constraint and the iso-value curves from the mean-standard deviation analysis can be written as

$$\begin{aligned} c_0 &= r_f + g_u + \frac{\mu}{\sigma}\sigma_c \\ c_0 &= \left[\left(\rho + (\gamma-1)g_e - \gamma(\gamma-1)\frac{\sigma_c^2}{2}\right)(1-\gamma)v\right]^{\frac{1}{1-\gamma}} \end{aligned}$$

We see that current-use gifts are equivalent to increasing the riskfree rate and therefore reduce risktaking. On the other hand, endowment gifts are equivalent to increasing the rate of time preference and therefore increase risktaking.

Geometric average model In the presence of gifts, the budget constraint and the geometric consumption rule become

$$dw_t = w_t dr_{p,t} + w_t(g_u + g_e) - c_t dt,$$

$$c_t dt = w_t\left(E_t dr_{p,t} + g_u - \frac{1}{2}\alpha^2\sigma^2 dt\right) = w_t\left(r_f + g_u + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right) dt.$$

We substitute the consumption rule into the budget constraint to obtain

$$\begin{aligned} dw_t &= w_t(r_f + \alpha\mu) + w_t\alpha\sigma dZ_t + w_t(g_u + g_e) - w_t\left(r_f + g_u + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right) dt \\ &= w_t\left(g_e + \frac{1}{2}\alpha^2\sigma^2\right) dt + w_t\alpha\sigma dZ_t \end{aligned}$$

The process for log consumption coincides with the process for log wealth

$$d\log(w_t) = d\log(c_t) = g_e dt + \alpha\sigma dZ_t$$

such that the portfolio constraint and the iso-value curves from the mean-standard deviation analysis can be written as

$$\begin{aligned} c_0 &= r_f + g_u + \frac{\mu}{\sigma}\sigma_c \\ c_0 &= \left[\left(\rho + (\gamma-1)g_e - (\gamma-1)^2\frac{\sigma_c^2}{2}\right)(1-\gamma)v\right]^{\frac{1}{1-\gamma}} \end{aligned}$$

The effects of gifts are exactly the same as in the arithmetic average model.

3.3 A Nominal Spending Constraint with Inflation

Consider a price level p_t following $dp_t = p_t \pi dt$ where π is inflation rate. The nominal rate becomes $r_f^\$ = r_f + \pi$ and the nominal return on the risky asset $dr_t^\$ = (r_f + \pi + \mu)dt + \sigma dZ_t$.

Arithmetic average model Suppose that the investor has a nominal sustainable spending constraint

$$c_t^\$ dt = w_t^\$ E[dr_{p,t}^\$]$$

where $c_t^\$ = c_t p_t$ and $w_t^\$ = w_t p_t$ so that

$$c_t dt = w_t E[dr_{p,t}^\$] = w_t(r_f^\$ + \alpha\mu)dt$$

The law of motion for nominal wealth is then

$$\begin{aligned}\frac{dw_t^\$}{w_t^\$} &= \alpha dr_t^\$ + (1-\alpha)r_f^\$ dt - \frac{c_t^\$}{w_t^\$}dt \\ &= \alpha(r_f^\$ + \mu)dt + \alpha\sigma dZ_t + (1-\alpha)r_f^\$ dt - E[dr_{p,t}^\$] \\ &= \alpha\sigma dZ_t\end{aligned}$$

This implies that real wealth follows

$$\frac{dw_t}{w_t} = \frac{dw_t^\$}{w_t^\$} - \pi dt = -\pi dt + \alpha\sigma dZ_t$$

Log consumption then follows

$$\begin{aligned}d\log(c_t) &= d\log(w_t) + d\log(E[dr_{p,t}^\$]) \\ &= \underbrace{\left(-\pi - \frac{\alpha^2\sigma^2}{2}\right)}_{\mu_c} dt + \underbrace{\alpha\sigma}_{\sigma_c} dZ_t\end{aligned}$$

We can now rewrite the portfolio constraint and iso-value curves as

$$\begin{aligned}c_0 &= r_f + \pi + \frac{\mu}{\sigma}\sigma_c \\ c_0 &= \left[\left(\rho - (\gamma-1)\pi - \gamma(\gamma-1)\frac{\sigma_c^2}{2}\right)(1-\gamma)v\right]^{\frac{1}{1-\gamma}}\end{aligned}$$

This shows that a nominal spending rule with positive inflation acts as a higher riskfree rate **and** a lower rate of time preference. Both reduce risktaking so that inflation also reduces risktaking.

Geometric average model Now the spending rule is

$$c_t^{\$}dt = w_t^{\$}E[d\log V_t^{\$}],$$

where $V_t^{\$}$ is defined as the solution to

$$\frac{dV_t^{\$}}{V_t^{\$}} = (r_f^{\$} + \alpha\mu)dt + \alpha\sigma dZ_t,$$

so that

$$c_t^{\$}dt = w_t^{\$}\left(r_f^{\$} + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)dt.$$

The law of motion for nominal wealth is

$$\begin{aligned}
\frac{dw_t^{\$}}{w_t^{\$}} &= \alpha dr_t^{\$} + (1-\alpha)r_f^{\$}dt - \frac{c_t^{\$}}{w_t^{\$}}dt \\
&= (r_f^{\$} + \mu)dt + \alpha\sigma dZ_t + (1-\alpha)r_f^{\$}dt - \left(r_f^{\$} + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)dt \\
&= \frac{1}{2}\alpha^2\sigma^2 dt + \alpha\sigma dZ_t
\end{aligned}$$

This implies the following process for log consumption

$$\begin{aligned}
d\log(c_t) &= d\log(w_t) \\
&= \underbrace{-\pi}_{\mu_c}dt + \underbrace{\alpha\sigma}_{\sigma_c}dZ_t
\end{aligned}$$

We can now rewrite the portfolio constraint and iso-value curves as

$$\begin{aligned}
c_0 &= r_f + \pi + \frac{\mu}{\sigma}\sigma_c - \frac{1}{2}\alpha^2\sigma^2 \\
c_0 &= \left[\left(\rho - (\gamma-1)\pi - (\gamma-1)^2\frac{\sigma_c^2}{2}\right)(1-\gamma)v\right]^{\frac{1}{1-\gamma}}
\end{aligned}$$

Inflation enters in the same way as it did for the arithmetic average model.

3.4 Epstein-Zin Preferences

In this section we show how to extend the model with a sustainably spending agent to Epstein-Zin utility. Most importantly, we show that all results derived for power utility still hold regardless of the Elasticity of Intertemporal Substitution ψ.

Lifetime value V_t for the class of recursive preferences is defined as the solution to

$$V_t = E_t \int_t^T f(c_s, V_s) ds$$

where $f(\cdot, \cdot)$ is the aggregator function. If wealth follows

$$dw_t = \mu(w_t)dt + \sigma(w_t)dZ_t,$$

the HJB equation is

$$0 = \max_{\alpha,c} f(c, V) + \frac{\partial V}{\partial W} \cdot \mu(w_t) + \frac{1}{2}\frac{\partial^2 V}{\partial W^2} \cdot \sigma(w_t)^2 \tag{A.8}$$

Epstein-Zin aggregator is defined as

$$f(c, V) = \frac{1}{1-\psi^{-1}} \left[\frac{\rho c^{1-\psi^{-1}}}{((1-\gamma)V)^{\frac{\gamma-\psi^{-1}}{1-\gamma}}} - \rho(1-\gamma)V \right] \tag{A.9}$$

3.4.1 Arithmetic Average Model

We now proceed to solving for the risky share for an agent with Epstein-Zin utility and an arithmetic sustainable spending constraint. Our approach will differ from the case of power utility. First, we conjecture a value function $V(w) = A\frac{w^{1-\gamma}}{1-\gamma}$. Second, we use the FOC to express A as a function of all other variables. Third, substitute A back into the HJB equation to solve for α.

Under the arithmetic sustainable spending constraint consumption is $c = w(r_f + \alpha\mu)$ and wealth follows

$$dw_t = w_t \alpha\sigma dZ_t \Rightarrow \mu(w_t) = 0, \sigma(w_t) = w_t \alpha\sigma.$$

Substituting $c, \mu(w_t)$ and $\sigma(w_t)$ along with our guess for V into the HJB equation we get

$$0 = \max_{\alpha} \left\{ \frac{1}{1-\psi^{-1}} \left[\frac{\rho(w(r_f + \alpha\mu))^{1-\psi^{-1}}}{(Aw^{1-\gamma})^{\frac{\gamma-\psi^{-1}}{1-\gamma}}} - \rho A w^{1-\gamma} \right] - \frac{1}{2}\gamma A w^{1-\gamma}\alpha^2\sigma^2 \right\}$$

we can factor $Aw^{1-\gamma}$ out of the maximization problem

$$0 = Aw^{1-\gamma} \max_{\alpha} \left\{ \frac{1}{1-\psi^{-1}} \left[\frac{\rho(r_f + \alpha\mu)^{1-\psi^{-1}}}{A^{\frac{1-\psi^{-1}}{1-\gamma}}} - \rho \right] - \gamma \frac{1}{2} \alpha^2 \sigma^2 \right\}.$$

Next, we use the first-order condition for α to express $1/A^{\frac{1-\psi^{-1}}{1-\gamma}}$ as a funcion of other parameters

$$\rho \frac{(r_f + \alpha\mu)^{-\psi^{-1}} \mu}{A^{\frac{1-\psi^{-1}}{1-\gamma}}} - \gamma\alpha\sigma^2 = 0 \implies \frac{1}{A^{\frac{1-\psi^{-1}}{1-\gamma}}} = \frac{\gamma\alpha\sigma^2}{\rho(r_f + \alpha\mu)^{-\psi^{-1}}\mu},$$

and substitute it back into the maximized HJB

$$0 = \frac{1}{1-\psi^{-1}} \left[\frac{(r_f + \alpha\mu)\gamma\alpha\sigma^2}{\mu} - \rho \right] - \gamma \frac{1}{2} \alpha^2 \sigma^2.$$

Rearrange to get a quadratic equation in α

$$\gamma\sigma^2 \frac{1+\psi^{-1}}{2(1-\psi^{-1})} \alpha^2 + \frac{\gamma\sigma^2 r_f}{(1-\psi^{-1})\mu} \alpha - \frac{\rho}{1-\psi^{-1}} = 0 \;\Big|\; \cdot \frac{(1-\psi^{-1})\mu}{\gamma\sigma^2},$$

$$\frac{\mu(1+\psi^{-1})}{2} \alpha^2 + r_f \alpha - \frac{\rho\mu}{\gamma\sigma^2} = 0,$$

That can be solved explicitly as

$$\alpha = \frac{-r_f + \sqrt{L}}{\mu(1+\psi^{-1})}, \; L = r_f^2 + 2\rho \frac{1+\psi^{-1}}{\gamma} \left(\frac{\mu}{\sigma} \right)^2$$

where we chose a positive solution for α. First note that when $\gamma = \psi^{-1}$ this solution coincides with the risky share for the power utility agent. Next, note that r_f, ρ, μ and σ enter in exactly the same way as for the solution to the power utility problem. Therefore, they have the same effect on the risky share. Using this closed form expression, one can show that the optimal risky share is increasing in ψ and $\lim_{\psi \to 0} \alpha = 0$.

3.4.2 Geometric Average Model

Next we derive comparative statics for an agent with Epstein-Zin utility and a geometric sustainable spending constraint. Similarly to the analysis of a power utility agent we are not able to derive a closed form expression for the risky share. However, we will be able to characterize the solution using two equations: (1) HJB equation and (2) FOC for α. We make the same guess that $V(w) = A\frac{w^{1-\gamma}}{1-\gamma}$

where A is an unknown constant, different from before. Under the geometric sustainable spending constraint consumption is $c_t = w_t\left(r_f + \alpha\mu + \frac{1}{2}\alpha^2\sigma^2\right)$ and wealth follows

$$dw_t = \frac{1}{2}w_t\alpha\sigma dt + w_t\alpha\sigma dZ_t \Rightarrow \mu(w_t) = \frac{1}{2}w_t\alpha\sigma, \sigma(w_t) = w_t\alpha\sigma.$$

Substituting $c, \mu(w_t)$ and $\sigma(w_t)$ along with our guess for V into the HJB equation we get

$$0 = Aw^{1-\gamma}\max_{\alpha}\left\{\frac{1}{1-\psi^{-1}}\left[\frac{\rho\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{1-\psi^{-1}}}{A^{\frac{1-\psi^{-1}}{1-\gamma}}} - \rho\right] + \frac{1}{2}(1-\gamma)\alpha^2\sigma^2\right\}$$

Before going further consider a limiting case when $\psi \to 0 \Rightarrow \psi^{-1} \to \infty$. Then

$$\frac{1}{1-\psi^{-1}} \to 0, \left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{1-\psi^{-1}} \to 0, A^{\frac{1-\psi^{-1}}{1-\gamma}} \to \infty$$

Therefore, the whole first term goes to zero leaving us with

$$0 = Aw^{1-\gamma}\max_{\alpha}\left\{-\frac{1}{2}(\gamma-1)\alpha^2\sigma^2\right\}$$

which result in the optimal portfolio rule $\alpha = 0$. Notice, however, that since consumption should be positive the limiting case will only have a solution when $r_f > 0$.

Now return to the HJB equation. The first-order condition is

$$\rho\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{-\psi^{-1}}(\mu - \alpha\sigma^2) + A^{\frac{1-\psi^{-1}}{1-\gamma}}(1-\gamma)\alpha\sigma^2 = 0 \tag{A.10}$$

This allows to express $A^{\frac{1-\psi^{-1}}{1-\gamma}}$ and substitute it back into the HJB equation to get

$$\frac{1}{1-\psi^{-1}}\left[\frac{\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)(\gamma-1)\alpha\sigma^2}{\mu - \alpha\sigma^2} - \rho\right] - \frac{1}{2}(\gamma-1)\alpha^2\sigma^2 = 0$$

We rearrange it and define

$$h \equiv \left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right) - \rho\frac{\mu - \alpha\sigma^2}{(\gamma-1)\alpha\sigma^2} - \frac{1}{2}\alpha(\mu - \alpha\sigma^2)(1-\psi^{-1}) = 0 \tag{A.11}$$

First, since consumption $c = r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2 > 0$ is positive, equation (A.11) implies that

$$\rho\frac{\mu - \alpha\sigma^2}{(\gamma-1)\alpha\sigma^2} + \frac{1}{2}\alpha(\mu - \alpha\sigma^2)(1-\psi^{-1}) > 0$$

$$\rho > -\frac{1}{2}\alpha^2\sigma^2(1-\psi^{-1})(\gamma-1) \tag{A.12}$$

We next utilize the implicit function theorem that says

$$\frac{d\alpha}{d\beta} = -\frac{\partial h/\partial\beta}{\partial h/\partial\alpha}$$

where β is any parameter, for example, the risk free rate r_f. We first sign $\frac{\partial h}{\partial \alpha}$:

$$\begin{aligned}
\frac{\partial h}{\partial \alpha} &= (\mu - \alpha\sigma^2) - \rho\frac{-\sigma^2(\gamma-1)\alpha\sigma^2 - (\gamma-1)\sigma^2(\mu-\alpha\sigma^2)}{[(\gamma-1)\alpha\sigma^2]^2} - \frac{1}{2}(1-\psi^{-1})(\mu - 2\alpha\sigma^2) \\
&= (\mu-\alpha\sigma^2) + \frac{\rho\mu}{(\gamma-1)\alpha^2\sigma^2} - \frac{1}{2}(1-\psi^{-1})(\mu-\alpha\sigma^2) + \frac{1}{2}(1-\psi^{-1})\alpha\sigma^2 \\
&= \left(1 - \frac{1}{2}(1-\psi^{-1})\right)(\mu-\alpha\sigma^2) + \frac{\rho\mu}{(\gamma-1)\alpha^2\sigma^2} + \frac{1}{2}(1-\psi^{-1})\alpha\sigma^2 \\
&= \frac{1}{2}(1+\psi^{-1})(\mu-\alpha\sigma^2) + \frac{\mu}{(\gamma-1)\alpha^2\sigma^2}\rho + \frac{1}{2}(1-\psi^{-1})\alpha\sigma^2
\end{aligned}$$

Next, use the inequality from (A.12)

$$\begin{aligned}
\frac{\partial h}{\partial \alpha} &> \frac{1}{2}(1+\psi^{-1})(\mu-\alpha\sigma^2) - \frac{\mu}{(\gamma-1)\alpha^2\sigma^2}\frac{1}{2}\alpha^2\sigma^2(1-\psi^{-1})(\gamma-1) + \frac{1}{2}(1-\psi^{-1})\alpha\sigma^2 \\
&= \frac{1}{2}(1+\psi^{-1})(\mu-\alpha\sigma^2) - \mu\frac{1}{2}(1-\psi^{-1}) + \frac{1}{2}(1-\psi^{-1})\alpha\sigma^2 \\
&= \frac{1}{2}(1+\psi^{-1})(\mu-\alpha\sigma^2) - \frac{1}{2}(\mu-\alpha\sigma^2)(1-\psi^{-1}) \\
&= \psi^{-1}(\mu-\alpha\sigma^2) > 0
\end{aligned}$$

Therefore, we have

$$\frac{\partial h}{\partial \alpha} > 0 \implies \frac{d\alpha}{d\beta} \propto -\frac{\partial h}{\partial \beta}$$

We next use this simplified expression to sign comparative statics.

Risk Free Rate The effect of the riskfree rate on the risky share

$$\frac{d\alpha}{dr_f} \propto -\frac{\partial h}{\partial r_f} = -1 < 0$$

Discount Rate The effect of the discount rate on the risky share

$$\frac{d\alpha}{d\rho} \propto -\frac{\partial h}{\partial \rho} = \frac{\mu-\alpha\sigma^2}{(\gamma-1)\alpha\sigma^2} > 0$$

Risk Premium Similarly to the analysis of a power utility agent, the effect of the risk premium on the risky share depends on the value of risk free rate and, in particular, there is a value of the risk free rate such that the risky share doesn't depend on the risk premium.

First, collect all terms with μ in equation (A.11)

$$h = \left(r_f - \frac{1}{2}\alpha^2\sigma^2\right) + \frac{\rho}{\gamma - 1} + \frac{1}{2}\alpha^2\sigma^2(1 - \psi^{-1}) + \mu\left[\alpha - \frac{\rho}{(\gamma - 1)\alpha\sigma^2} - \frac{1}{2}\alpha(1 - \psi^{-1})\right] = 0$$

Risky share α doesn't change with the risk premium μ when $\alpha = \alpha^*$ such that the expression in the brackets is exactly zero

$$\alpha^* - \frac{\rho}{(\gamma - 1)\alpha^*\sigma^2} - \frac{1}{2}\alpha^*(1 - \psi^{-1}) = 0$$

$$(\alpha^*)^2 = \frac{2\rho}{(\gamma - 1)(\psi^{-1} + 1)\sigma^2}$$

To find the risk free rate that implies $\alpha = \alpha^*$, express r_f from h and substitute $(\alpha^*)^2$

$$\begin{aligned}
r_f^* &= \frac{\psi^{-1}}{2}(\alpha^*)^2\sigma^2 - \frac{\rho}{\gamma - 1} \\
&= \frac{\psi^{-1}}{2}\frac{2\rho}{(\gamma - 1)(\psi^{-1} + 1)\sigma^2}\sigma^2 - \frac{\rho}{\gamma - 1} \\
&= \frac{\psi^{-1}\rho}{(\gamma - 1)(\psi^{-1} + 1)} - \frac{\rho}{\gamma - 1} \\
&= \frac{\rho}{\gamma - 1}\left(\frac{\psi^{-1}}{(\psi^{-1} + 1)} - 1\right) \\
&= -\frac{\rho}{(\gamma - 1)(\psi^{-1} + 1)} < 0
\end{aligned}$$

Where we used the assumption $\gamma > 1$. When $r_f > r_f^*$, the risk premium has a standard effect on the risky share. When $r_f < r_f^*$, the effect is reversed: a higher risk premium leads to a lower risky share.

Elasticity of Intertemporal Substitution The effect of EIS ψ on the risky share is

$$\frac{d\alpha}{d\psi} \propto -\frac{\partial h}{\partial \psi} = \frac{1}{2}\alpha(\mu - \alpha\sigma^2)\psi^{-2} > 0$$

Higher Elasticity of Intertemporal Substitution leads to a larger risky share.

3.5 Equilibrium in the Risky Asset Market

In this section, we derive existence conditions for the equilibrium in the risky asset market and derive the relationship between an exogenous risk free rate and the risk premium in equilibrium.

Geometric Constraint We start by looking at $\gamma > 1$ case and describe the $\gamma < 1$ case below. We have two existence conditions related to the existence of a solution to the partial equilibrium problem. When fully invested in the risky asset the lifetime value of the agent should converge. For $\alpha = 1$, the denominator in (A.3) is positive when

$$\rho > \frac{1}{2}\sigma^2(\gamma - 1)^2. \tag{A.13}$$

Second, when fully invested in the risky asset, the agent should have positive consumption

$$c = r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2 = [\alpha = 1] = r_f + \mu - \frac{1}{2}\sigma^2 > 0, \tag{A.14}$$

where μ is the risk premium that clears the market for the risky asset.

Finally, we need to ensure that it is possible to induce the agent to hold all his wealth in the risky asset by adjusting the risk free rate. As discussed in the main text this requires $\alpha^* > 1$ for $r_f > r_f^*$ and $\alpha^* < 1$ for $r_f < r_f^*$. Proof of Proposition 2 derives

$$\alpha^* = \sqrt{\frac{2\rho}{(\gamma^2 - 1)\sigma^2}}$$

Therefore, it is possible to clear the market for the risky asset by inducing $\alpha = 1$ for the agent with a sustainable spending constraint if

$$\begin{bmatrix} \rho > \frac{1}{2}\sigma^2(\gamma^2 - 1) \text{ when } r_f > r_f^* \\ \rho < \frac{1}{2}\sigma^2(\gamma^2 - 1) \text{ when } r_f < r_f^* \end{bmatrix} \tag{A.15}$$

We also know from the proof of Proposition 2 that

$$r_f^* = -\frac{\rho}{\gamma^2 - 1}$$

so that (A.15) implies

$$\begin{bmatrix} -\frac{\rho}{\gamma^2 - 1} < -\frac{1}{2}\sigma^2 \text{ when } r_f > r_f^* \\ -\frac{\rho}{\gamma^2 - 1} > -\frac{1}{2}\sigma^2 \text{ when } r_f < r_f^* \end{bmatrix} \Rightarrow \begin{bmatrix} r_f^* < -\frac{1}{2}\sigma^2 \text{ when } r_f > r_f^* \\ r_f^* > -\frac{1}{2}\sigma^2 \text{ when } r_f < r_f^* \end{bmatrix} \tag{A.16}$$

Keeping these existence conditions in mind, we now proceed to solving for the risk premium that will clear the market for risky asset as a function of an exogenous risk free rate. The optimal risky share α is the solution to maximization problem (A.3) with the following first-order condition

$$(\mu-\alpha\sigma^2)(1-\gamma)\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2\right)^{-\gamma}\left(\rho-\frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2\right)+(1-\gamma)^2\alpha\sigma^2\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2\right)^{1-\gamma}=0,$$

$$(\mu-\alpha\sigma^2)\left(\rho-\frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2\right)+(1-\gamma)\alpha\sigma^2\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2\right)=0.$$

Imposing $\alpha = 1$ we obtain the following expression relating the risk free rate r_f and the risk premium μ in an affine way:

$$(\mu-\sigma^2)\left(\rho-\frac{1}{2}(1-\gamma)^2\sigma^2\right)+(1-\gamma)\sigma^2\left(r_f+\mu-\frac{1}{2}\sigma^2\right)=0 \tag{A.17}$$

Solving for μ as a function of r_f we obtain

$$\mu=\sigma^2\left[\frac{\rho-\frac{1}{2}\sigma^2(\gamma^2-\gamma)}{\rho-\frac{1}{2}\sigma^2(\gamma^2-1)}+\frac{\gamma-1}{\rho-\frac{1}{2}\sigma^2(\gamma^2-1)}r_f\right] \tag{A.18}$$

Subsituting (A.18) into condition (A.15) we get

$$\frac{\left(r_f+\frac{\sigma^2}{2}\right)\left(\rho-\frac{1}{2}\sigma^2(\gamma-1)^2\right)}{\left(\rho-\frac{1}{2}\sigma^2(\gamma^2-1)\right)}>0$$

Combining this with condition (A.13) we obtain condition (A.15) expressed in terms of exogenous parameters:

$$\begin{bmatrix} r_f>-\frac{1}{2}\sigma^2 \text{ when } r_f>r_f^* \\ r_f<-\frac{1}{2}\sigma^2 \text{ when } r_f<r_f^* \end{bmatrix} \tag{A.19}$$

Combining all existence conditions (A.13), (A.15), (A.16) and (A.19) together, when $\gamma > 1$, there exists a level of risk premium μ the clears the market for risky asset defined in (A.18) when

$$\begin{bmatrix} \text{Case 1:} \quad r_f^*<-\frac{\sigma^2}{2}<r_f \text{ and } \rho>\frac{\sigma^2}{2}(\gamma^2-1) \\ \text{Case 2:} \quad r_f<-\frac{\sigma^2}{2}<r_f^* \text{ and } \frac{\sigma^2}{2}(\gamma^2-1)<\rho<\frac{\sigma^2}{2}(\gamma^2-1) \end{bmatrix} \tag{A.20}$$

It is easy to see that under these existence conditions, the risk premium is increasing in the risk free rate when $r_f > r_f^*$ and is decreasing in the risk free rate when $r_f < r_f^*$.

Equilibrium for $\gamma < 1$ When $\gamma < 1$, partial equilibrium risky share always increases in the risk premium and, as a result, there is no subtle issue with upper and lower bounds for α. Thus, the only conditions left to ensure the existence of a general equilibrium are the ones that ensure the existence of the solution to the partial equilibrium problem. First, the lifetime value of the agent when fully invested in the risky asset should converge. Similarly to before this requires

$$\rho > \frac{1}{2}\sigma^2(\gamma - 1)^2 \tag{A.21}$$

Second, the portfolio constraint should intersect the x-axis to the left of the "pinned" point where the indifference curve is equal to zero, requiring

$$\frac{\mu}{\sigma} + \sqrt{\left(\frac{\mu}{\sigma}\right)^2 + 2r_f} < \sqrt{\frac{2\rho}{(\gamma - 1)^2}} \tag{A.22}$$

where μ is equal to (A.18). Under conditions (A.21) and (A.22) the equilibrium risk premium as a function of the risk free rate is given in equat (A.18).

Arithmetic Constraint Below we derive the relationship between an exogenous risk free rate and risk premium in equilibrium for the risky asset when the agent follows an arithmetic as opposed to geometric sustainable spending rule.

Unlike the geometric constraint, the arithmetic constraint allows to solve for the risky share in closed form:

$$\alpha = \frac{-r_f + \sqrt{K}}{\mu(1+\gamma)} \text{ where } K = r_f^2 + 2\rho\left(\frac{1+\gamma}{\gamma}\right)\left(\frac{\mu}{\sigma}\right)^2$$

Similarly to the geometric constraint, there is a level of riskfree rate where the optimal risky share in the partial equilibrium solution doesn't depend on the risk premium. This point doesn't depend on parameters and is equal to $r_f^* = 0$. Risky share at this point is

$$\alpha^* = \frac{1}{\sigma}\sqrt{\frac{2\rho}{\gamma(\gamma+1)}}$$

A general equilibrium exists if under full investment in the risky asset the lifetime value converges, i.e.

$$\begin{bmatrix} \rho > \frac{1}{2}\sigma^2\gamma(\gamma+1) \text{ for } r_f > 0 \\ \rho < \frac{1}{2}\sigma^2\gamma(\gamma+1) \text{ for } r_f < 0 \end{bmatrix} \tag{A.23}$$

Finally, for an equilibrium to exist full investment in the risky asset should provide the agent with positive consumption, i.e. $r_f + \mu > 0$ where μ is the risk premium that clears the market for the risky asset.

By equating the closed form solution for the risky share (2.2) to one we can solve for μ as

$$\mu = \frac{\gamma\sigma^2}{\rho - \frac{1}{2}\sigma^2\gamma(\gamma+1)} \times r_f$$

Hence, for $r_f > 0$, μ is increasing in r_f and for $r_f < 0$, μ is decreasing in r_f consistent with the analysis of partial equilibrium.

We can use this expression for μ to derive the condition on parameters that guarantees positive consumption for the sustainably spending agent inn general equilibrium

$$r_f + \frac{\gamma\sigma^2}{\rho - \frac{1}{2}\sigma^2\gamma(\gamma+1)} \times r_f > 0$$

$$\frac{\rho - \frac{1}{2}\sigma^2\gamma(\gamma-1)}{\rho - \frac{1}{2}\sigma^2\gamma(\gamma+1)} \times r_f > 0$$

This is always satisfied if (A.23) is satisfied.

4 A Dynamic Model

This section presents the approach for solving the dynamic model presented in the main text.

HJB Equation for Multiple States We first show the general form of an HJB equation with multiple dynamic constraints. The general problem is

$$\begin{aligned} v(x_0) &= \max_{c_t} E\int_0^\infty e^{-\rho t} u(x_t, c_t)dt \\ dx_t &= f(x_t, c_t)dt + \sigma(x_t, c_t)dZ_t \end{aligned}$$

where x_t is $N \times 1$, dZ_t is $M \times 1$ and c_t is $K \times 1$. First, we need to define a $N \times N$ matrix

$$\Sigma(x_t, c_t) = \sigma(x_t, c_t)\sigma(x_t, c_t)'$$

Using $\Sigma(x_t, c_t)$ we can write the HJB equation as

$$\rho v(x) = \max_c \left\{ u(x,c) + \sum_{i=1}^N \frac{\partial v}{\partial x_i} f(x,c) + \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 v}{\partial x_i \partial x_j}\Sigma_{ij}(x,c) \right\}$$

where $\Sigma_{ij}(x,c)$ is (i,j) element of matrix $\Sigma(x,c)$.

The dynamic model presented in the main text has the following maximization problem

$$\max_{\alpha_t} E_0 \int_0^\infty e^{-\rho t} u(c_t)dt$$

$$\text{subject to } c_t = w_t\left(r_t + \alpha_t \mu - \frac{1}{2}\alpha_t^2\sigma^2\right)$$

$$\begin{pmatrix} dw_t \\ dr_t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}w_t\alpha_t^2\sigma^2 \\ \phi(r_t) \end{pmatrix} + \begin{pmatrix} w_t\alpha_t\sigma & 0 \\ \nu r_t \eta & \nu r_t\sqrt{1-\eta^2} \end{pmatrix} \begin{pmatrix} dZ_t^{(1)} \\ dZ_t^{(2)} \end{pmatrix}$$

Matrix Σ is

$$\Sigma \equiv \begin{pmatrix} w_t\alpha_t\sigma & 0 \\ \nu r_t\eta & \nu r_t\sqrt{1-\eta^2} \end{pmatrix} \begin{pmatrix} w_t\alpha_t\sigma & \nu r_t\eta \\ 0 & \nu r_t\sqrt{1-\eta^2} \end{pmatrix} = \begin{pmatrix} w_t^2\alpha_t^2\sigma^2 & w_t\alpha_t\sigma\nu r_t\eta \\ w_t\alpha_t\nu r_t\eta & \nu^2 r_t^2 \end{pmatrix}$$

We can use Σ, the general form of the HJB equation presented above and the guess for value function $v(w,r) = A(r)\frac{w^{1-\gamma}}{1-\gamma}$ to derive the HJB equation presented in the main text.

Problem Our goal is to numerically solve the following system of equations

$$\left(r + \alpha^*\mu - \frac{1}{2}(\alpha^*)^2\sigma^2\right)^{-\gamma}(\mu - \alpha^*\sigma^2) + A(r)(1-\gamma)\alpha^*\sigma^2 + A'(r)\sigma\nu r\eta = 0$$

$$\begin{aligned}\rho A(r)\frac{1}{1-\gamma} = \frac{\left(r + \alpha^*\mu - \frac{1}{2}(\alpha^*)^2\sigma^2\right)^{1-\gamma}}{1-\gamma} &+ A(r)\frac{1}{2}(1-\gamma)(\alpha^*)^2\sigma^2 + A'(r)\frac{1}{1-\gamma}\frac{1}{2}\nu^2 r \\ &+ \frac{1}{2}A''(r)\frac{1}{1-\gamma}\nu^2 r^2 + A'(r)\sigma\alpha^*\nu r\eta\end{aligned}$$

where the first equation is the FOC and the second equation is the maximized HJB equation, i.e. the HJB equation evaluated at the optimal risky share $\alpha = \alpha^*$ that can be derived from the FOC.

Discretization We first discretize the state space $r = r_1, \ldots, r_I$ with equidistant intervals such that $r_i - r_{i-1} = \Delta r \ \forall i$. To simplify notation denote $A(r_i) = A_i$. Denote the solution to the FOC for particular level of the interest rate r_i as α_i. We approximate the derivatives as follows

$$(A')_i \approx \frac{A_{i+1} - A_{i-1}}{2\Delta r}$$

$$(A'')_i \approx \frac{A_{i+1} - 2A_i + A_{i-1}}{(\Delta r)^2}$$

With these approximations the FOC becomes

$$0 = \left(r_i + \alpha_i\mu - \frac{1}{2}\alpha_i^2\sigma^2\right)^{-\gamma}(\mu - \alpha_i\sigma^2) + A_i(1-\gamma)\alpha_i\sigma^2 + \frac{A_{i+1} - A_{i-1}}{2\Delta r}\sigma\nu r_i\eta, \tag{A.24}$$

and the discretized HJB equation becomes

$$\begin{aligned}\rho A_i = \left(r_i + \alpha_i\mu - \frac{1}{2}\alpha_i^2\sigma^2\right)^{1-\gamma} &+ A_i\frac{1}{2}(1-\gamma)^2\alpha_i^2\sigma^2 + \frac{A_{i+1} - A_{i-1}}{2\Delta r}\frac{1}{2}\nu^2 r \\ &+ \frac{1}{2}\frac{A_{i+1} - 2A_i + A_{i-1}}{(\Delta r)^2}\nu^2 r_i^2 + \frac{A_{i+1} - A_{i-1}}{2\Delta r}(1-\gamma)\alpha_i\sigma r_i\nu\eta,\end{aligned}$$

where we multiplied the whole expression by $1-\gamma$. Collecting the A terms we get

$$\begin{aligned}\rho A_i = \left(r_i + \alpha_i\mu - \frac{1}{2}\alpha_i^2\sigma^2\right)^{1-\gamma} &+ A_{i-1}\left[\frac{\nu^2 r_i^2}{2(\Delta r)^2} - \frac{(1-\gamma)\alpha_i\sigma r_i\nu\eta}{2\Delta r} - \frac{1}{2\Delta r}\frac{\nu^2 r_i}{2}\right] + A_i\left[\frac{1}{2}(1-\gamma)^2\alpha_i^2\sigma^2 - \frac{\nu^2 r_i^2}{(\Delta r)^2}\right] \\ &+ A_{i+1}\left[\frac{\nu^2 r_i^2}{2(\Delta r)^2} + \frac{(1-\gamma)\alpha_i\sigma r_i\nu\eta}{2\Delta r} + \frac{1}{2\Delta r}\frac{\nu^2 r_i}{2}\right]\end{aligned}$$

We impose the "reflecting barrier" constraints $A_0 = A_1, A_{I+1} = A_I$. Under these constraints the equation for $i = 1$ and $i = I$ becomes

$$\begin{aligned}
\rho A_1 = \left(r_1 + \alpha_1\mu - \frac{1}{2}\alpha_1^2\sigma^2\right)^{1-\gamma} &+ A_1\left[\frac{1}{2}(1-\gamma)^2\alpha_1^2\sigma^2 - \frac{\nu^2 r_1^2}{2(\Delta r)^2} - \frac{(1-\gamma)\alpha_1\sigma r_1\nu\eta}{2\Delta r} - \frac{1}{2\Delta r}\frac{\nu^2 r_1}{2}\right] \\
&+ A_2\left[\frac{\nu^2 r_1^2}{2(\Delta r)^2} + \frac{(1-\gamma)\alpha_1\sigma r_1\nu\eta}{2\Delta r} + \frac{1}{2\Delta r}\frac{\nu^2 r_1}{2}\right]
\end{aligned}$$

$$\begin{aligned}
\rho A_I = \left(r_I + \alpha_I\mu - \frac{1}{2}\alpha_I^2\sigma^2\right)^{1-\gamma} &+ A_{I-1}\left[\frac{\nu^2 r_I^2}{2(\Delta r)^2} - \frac{(1-\gamma)\alpha_I\sigma r_I\nu\eta}{2\Delta r} - \frac{1}{2\Delta r}\frac{\nu^2 r_I}{2}\right] \\
&+ A_I\left[\frac{1}{2}(1-\gamma)^2\alpha_I^2\sigma^2 - \frac{\nu^2 r_I^2}{2(\Delta r)^2} + \frac{(1-\gamma)\alpha_I\sigma r_I\nu\eta}{2\Delta r} + \frac{1}{2\Delta r}\frac{\nu^2 r_I}{2}\right]
\end{aligned}$$

Now we write this in matrix notation to get

$$\begin{aligned}
x_i &= \frac{\nu^2 r_i^2}{2(\Delta r)^2} - \frac{(1-\gamma)\alpha_i\sigma r_i\nu\eta}{2\Delta r} - \frac{1}{2\Delta r}\frac{\nu^2 r_i}{2} \\
y_i &= \frac{1}{2}(1-\gamma)^2\alpha_i^2\sigma^2 - \frac{\nu^2 r_i^2}{(\Delta r)^2} \\
z_i &= \frac{\nu^2 r_i^2}{2(\Delta r)^2} + \frac{(1-\gamma)\alpha_i\sigma r_i\nu\eta}{2\Delta r} + \frac{1}{2\Delta r}\frac{\nu^2 r_i}{2}
\end{aligned}
\Rightarrow B^n = \begin{pmatrix}
y_1 + x_1 & z_1 & 0 & 0 & \cdots \\
x_2 & y_2 & z_2 & 0 & \cdots \\
0 & x_3 & y_3 & z_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots \\
0 & \cdots & 0 & x_I & y_I + z_I
\end{pmatrix}$$

where n denotes the iteration step and B^n emphasizes that it is calculated using α_i^n that is itself calculated using $\mathbf{A}^n$. Using this notation we can write the iteration as

$$\frac{\mathbf{A}^{n+1} - \mathbf{A}^n}{\Delta} + \rho\mathbf{A} = \mathbf{u}^n + B^n\mathbf{A}$$

The explicit method is

$$\frac{\mathbf{A}^{n+1} - \mathbf{A}^n}{\Delta} + \rho\mathbf{A}^n = \mathbf{u}^n + B^n\mathbf{A}^n \Rightarrow \mathbf{A}^{n+1} = \mathbf{A}^n + \Delta\left(\mathbf{u}^n + B^n\mathbf{A}^n - \rho\mathbf{A}^n\right)$$

However, the **implicit** method has better convergence properties:

$$\frac{\mathbf{A}^{n+1} - \mathbf{A}^n}{\Delta} + \rho\mathbf{A}^{n+1} = \mathbf{u}^n + B^n\mathbf{A}^{n+1} \Rightarrow \mathbf{A}^{n+1} = \left(\left(\frac{1}{\Delta} + \rho\right)\mathrm{eye}(I) - B^n\right)^{-1}\left(\mathbf{u}^n + \frac{1}{\Delta}\mathbf{A}^n\right)$$

Even though this method requires matrix inversion at every step of the iteration, the matrix is sparse and can be inverted efficiently using appropriate routines (e.g. "\" command in Matlab or Julia uses an iterative solver allowing to avoid the inversion of the full matrix).

Numerical Algorithm The full algorithm

1. Given $\mathbf{A}^n$ numerically solve for a vector of α_i^n using the first-order condition in equation (A.24)
2. Given a vector α_i^n form vector $\mathbf{u}$ and matrix B^n
3. Update $\mathbf{A}$ using implicit scheme

$$\mathbf{A}^{n+1} = \left(\left(\frac{1}{\Delta} + \rho\right)\text{eye}(I) - B^n\right)^{-1}\left(\mathbf{u}^n + \frac{1}{\Delta}\mathbf{A}^n\right)$$

4. Iterate until the difference between $\mathbf{A}^n$ and $\mathbf{A}^{n+1}$ becomes small, say less than 10^{-6}.

Drift of Log Consumption The drift of log consumption in the dynamic model is

$$\frac{Ed\log(c)}{dt} = \frac{1}{2}\nu^2 r_{ft}\frac{f'(r_{ft})f(r) + f''(r_{ft})f(r_{ft}) - (f'(r_{ft}))^2}{(f(r_{ft}))^2} \tag{A.25}$$

where

$$\begin{aligned} f(r) &= r + \alpha(r)\mu - \frac{1}{2}\alpha(r)^2\sigma^2 \\ f'(r) &= 1 + \alpha'(r)\mu - \alpha(r)\alpha'(r)\sigma^2 \\ f''(r) &= 1 + \alpha''(r)\mu - \left((\alpha'(r))^2 + \alpha(r)\alpha''(r)\right)\sigma^2 \end{aligned}$$

Using the numerical solution for the risky share as a function of the riskfree rate $\alpha(r)$ we can evaluate (A.25). In Figure A.4 we show drift of log consumption as a function of ${r_f}$[2]. As we mention in the main text, the drift is not zero and is, in fact, negative. However, the absolute magnitude is small as can be seen from the y-axis.

Comparing Hedging Demand to Merton Model We numerically solve the same dynamic model for an unconstrained agent – the standard Merton model – for the same functional forms and parameters to compare the hedging demand for constrained and unconstrained agents. In Figure A.5 we plot hedging demand defined as the risky share for a specified correlation less the risky share for zero correlation. The solid lines show the hedging demand for an agent with a sustainable consumption constraint and the dashed lines show the hedging demand for an unconstrained agent. Even though the dashed and solid lines are not identical the difference is very small implying that the sustainable consumption constraint does not alter the hedging demand in a substantial way.

[2] We use the parameters from the baseline model in the main text and zero correlation

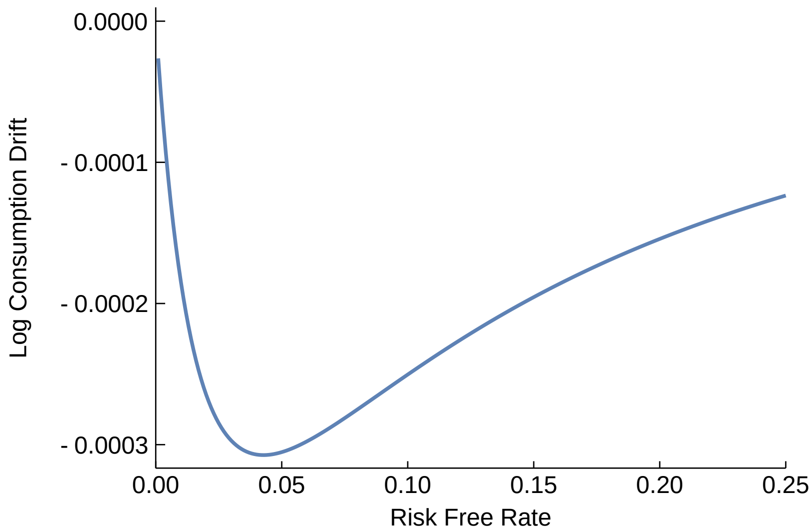

Figure A.4: Drift of log consumption in Dynamic Model

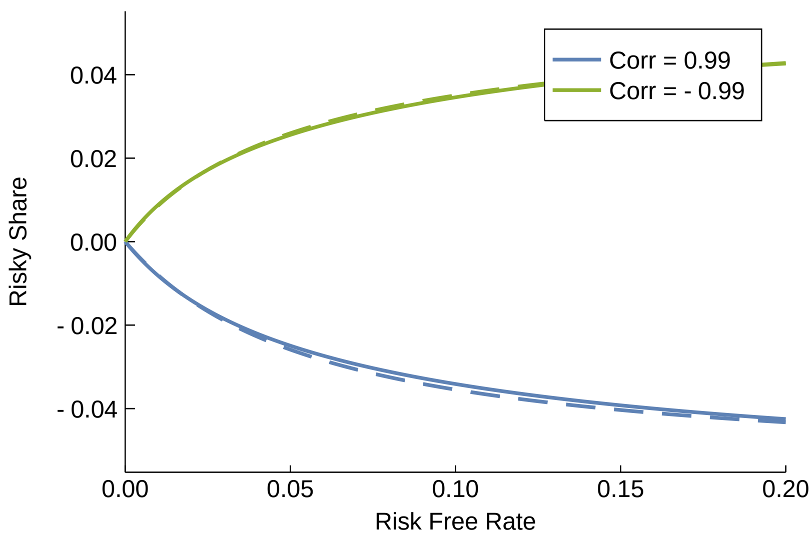

Figure A.5: Hedging Demand for Constrained (Solid) and Unconstrained (Dashed) Agents

References

Perold, Andre F., and Erik Stafford. "Harvard Management Company (2010)." Harvard Business School Case 211-004, September 2010. (Revised May 2012.)