Asset Pricing Notes. Chapter 9: Intertemporal Risk

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1 Introduction

In this chapter we are going to look at the portfolio choice problem of a long-term investor and ask a question: under what conditions does the allocation of a long term investor differs compared to the allocation of a myopic investor? We will use the intertemporal CAPM derived in chapter 9 that says that the SDF of an investor is

$$\tilde{m}_{t+1} = -\gamma r_{w,t+1} - (\gamma - 1) \underbrace{(E_{t+1} - E_t) \sum_{j=1}^{\infty} r_{w,t+1+j}}_{\tilde{h}_{t+1}}$$

so that the risk premium is

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = -\sigma_{imt} = \gamma \sigma_{iwt} + (\gamma - 1)\sigma_{iht}$$

This formula actually doesn't require Epstein-Zin preferences and it is still valid (up to approximation) with a power utility. We can see that since there is nothing preventing us from setting $\gamma = 1/\psi$ to get to the case of power utility.

2 Myopic Portfolio Choice

Definition 1 (Myopic Portoflio Choice)**.** *At a given point in time, investor has the same optimal portfolio regardless of investment horizon. Note that this doesn't mean that his optimal portfolio is constant over time.*

We're going to look at several examples where portoflio choice is myopic

Power Utility with $\gamma = 1$ Suppose that such investor maximizes wealth after two periods. He solves

$$\begin{aligned}
\max_p E_t \frac{W_{t+2}^{1-\gamma}}{1-\gamma} &\implies \max_p \log E_t \frac{W_{t+2}^{1-\gamma}}{1-\gamma} \\
&\implies \max_p E_t \log(W_{t+2}^{1-\gamma}) + \frac{1}{2} var_t(\log(W_{t+2}^{1-\gamma})) \\
&\implies \max_p (1-\gamma) E_t w_{t+2} + \frac{(1-\gamma)^2}{2} var_t(w_{t+2}) \\
&\implies \max_p E_t w_{t+2} + \frac{1-\gamma}{2} var_t(w_{t+2}) \\
&\implies \max_p E_t r_{p,t\to t+2} + \frac{1-\gamma}{2} var_t(r_{p,t\to t+2})
\end{aligned}$$

where we used log budget constraint without consumption $W_t(1 + R_{p,t\to t+2}) = W_{t+2} \implies w_t + r_{p,t+2} = w_{t+2}$. Log utility has $\gamma = 1$ so that the problem becomes

$$\max_p E_t r_{p,t\to t+2} = \max_p E_t r_{p,t+1} + E_t r_{p,t+2}$$

Given that the investor can rebalance portfolio weight at time t impact only the term $E_t r_{p,t+1}$. Therefore, at time t such investor maximizes $E_t r_{p,t+1}$ which is the same objective function as for log investor that invests for 1 period only.

Independent Returns Over Time When returns are independently distributed over time (but no necessarily identically) we can show that the portfolio choice is myopic over time. The argument is with backward induction. With power utility the value function takes form

$$V_t = \chi_t W_t^{1-\gamma}$$

where χ_t is the scale factor that reflects investment opportunities (state variables). Hence, at the last period $T-1$ the agent solve one period problem. At period $T-2$ he maximizes a function of wealth at period $T-1$ and so on. Since the returns are independent over time state variables are independent of decision made in previous periods. If moreover, the returns are identically distributed over time, the portfolio choice is constant.

To see how this works consider a power utility investor that faces iid lognormal return and maximizes his wealth in two periods. His problem is (from previous calculations)

$$\underbrace{\max_p r_{p,t,t+2} + \frac{1}{2} var_t(r_{p,t,t+2})}_{\text{average simple return}} - \frac{\gamma}{2} var_t(r_{p,t,t+2})$$

we can use the return approximation from chapter two $r_{p,t+1} - r_{f,t+1} = \alpha_t(r_{t+1} - r_{f,t+1}) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma^2$ and applying it for two periods

$$\begin{aligned}
r_{p,t,t+2} - 2r_f &= (r_{p,t+1} - r_f) + (r_{p,t+2} - r_f) \\
&= \alpha_t(r_{t+1} - r_f) + \alpha_{t+1}(r_{t+1} - r_f) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma^2 + \frac{1}{2}\alpha_{t+1}(1-\alpha_{t+1})\sigma^2 \\
var_t(r_{p,t,t+2}) &= (\alpha_t^2 + \alpha_{t+1}^2)\sigma^2
\end{aligned}$$

Mean simple return in this case

$$E_t r_{p,t,t+2} + \frac{1}{2} var_t(r_{p,t,t+2}) = 2r_f + (\alpha_t + \alpha_{t+1}) \left[Er - r_f + \frac{1}{2}\sigma^2 \right]$$

Average simple return depends only on the sum of $(\alpha_t + \alpha_{t+1}$. The whole problem becomes

$$\max_{\alpha_t, \alpha_{t+1}} r_{p,t,t+2} + \frac{1}{2} var_t(r_{p,t,t+2}) - \frac{\gamma}{2} var_t(r_{p,t,t+2}) = 2r_f + (\alpha_t + \alpha_{t+1}) \left[Er - r_f + \frac{1}{2}\sigma^2 \right] + (\alpha_t^2 + \alpha_{t+1}^2)\sigma^2$$

which is solved at $\alpha_t = \alpha_{t+1}$

3 Intertemporal Hedging

Consider a power utility investor maximizing his wealth in K periods and lognormal return not necessarily independently nor identically ditributed over time. Use this investor's problem from the previous part

$$\max_{\alpha_t} E_t r_{p,K,t+K} + \frac{1}{2}(1-\gamma)var_t(r_{p,K,t+K})$$

$$= \max_{\alpha_t} E_t[r_{p,t+1}] + \frac{1}{2}(1-\gamma)var_t(r_{p,t+1}) + [E_t r_{p,K,t+K} - E_t[r_{p,t+1}]] + \frac{1-\gamma}{2}\left[var_t(r_{p,K,t+K}) - var_t(r_{p,t+1})\right]$$

We can decompose K period return into $r_{p,K,t+K} = r_{p,t+1} + r_{p,K-1,t+K}$ and plug this into the problem above

$$\max_{\alpha_t} E_t[r_{p,t+1}] + \frac{1}{2}(1-\gamma)var_t(r_{p,t+1}) + E_t r_{p,K-1,t+K} + \frac{1-\gamma}{2}\left[var_t(r_{p,t+1} + r_{p,K-1,t+K}) - var_t(r_{p,t+1})\right]$$

$$\max_{\alpha_t} E_t[r_{p,t+1}] + \frac{1}{2}(1-\gamma)var_t(r_{p,t+1}) + E_t r_{p,K-1,t+K} + \frac{1-\gamma}{2}\left[var_t(r_{p,K-1,t+K}) + 2cov_t(r_{p,t+1}, r_{p,K-1,t+K}))\right]$$

when we allow for rebalancing we have that both $E_t r_{p,K-1,t+K}$ and $var_t(r_{p,K-1,t+K})$ are unaffected by the choice of α_t. Hence, we can write the problem as

$$\max_{\alpha_t} E_t[r_{p,t+1}] + \frac{1}{2}(1-\gamma)var_t(r_{p,t+1}) + (1-\gamma)cov_t(r_{p,t+1}, r_{p,K-1,t+K}))$$

subtract $r_{f,t+1}$ that is knows in advance and, therefore, doesn't affect the optimal choice and use return approximation from chapter 2

$$\max_{\alpha_t} [E_t r_{p,t+1} - r_{f,t+1}] + \frac{1}{2}(1-\gamma)var_t(r_{p,t+1}) + (1-\gamma)cov_t(r_{p,t+1}, r_{p,K-1,t+K}))$$

$$\max_{\alpha_t} \alpha_t(E_t r_{t+1} - r_{f,t+1}) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma_t^2 + \frac{1}{2}(1-\gamma)\alpha_t^2\sigma_t^2 + (1-\gamma)cov_t(r_{p,t+1}, r_{p,K-1,t+K})$$

First order condition of this problem is

$$\begin{aligned} 0 &= \alpha_t(E_t r_{t+1} - r_{f,t+1}) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma_t^2 + \frac{1}{2}(1-\gamma)\alpha_t^2\sigma_t^2 + (1-\gamma)cov_t(r_{p,t+1}, r_{p,K-1,t+K}) \\ &= (E_t r_{t+1} - r_{f,t+1}) + \frac{1}{2}(1-2\alpha_t)\sigma_t^2 + (1-\gamma)\alpha_t\sigma_t^2 + (1-\gamma)\frac{d}{d\alpha_t}cov_t(r_{p,t+1}, r_{p,K-1,t+K}) \\ \implies \gamma\alpha_t\sigma_t^2 &= E_t r_{t+1} - r_{f,t+1} + \frac{1}{2}\sigma_t^2 + (1-\gamma)\frac{d}{d\alpha_t}cov_t(r_{p,t+1}, r_{p,K-1,t+K}) \qquad (1) \\ \implies \alpha_t &= \underbrace{\frac{E_t r_{t+1} - r_{f,t+1} + \frac{1}{2}\sigma_t^2}{\gamma\sigma_t^2}}_{\text{standard term from chapter 2}} - \underbrace{\frac{\gamma-1}{\gamma\sigma_t^2}\cdot\frac{d}{d\alpha_t}cov_t(r_{p,t+1}, r_{p,K-1,t+K})}_{\text{intertemporal hedging term}} \end{aligned}$$

To understand this suppose that investment opportuinities in at period $t+1$ (expectation of returns going forward) are positively correlated with return at time $t+1$. This means that the agent has more wealth when investment opp. are good going forward and less wealth when investment opp. are bad going forward. This increases the volatility of two periods return. In this case, $\frac{d}{d\alpha_t}cov_t \propto \alpha_t$. This is penalized by a conservative investor with $\gamma > 1$ so that the asset weight of risky asset is lower in equation (1).

3.1 Hedging Interest Rates

Here we consider an agent with EZ utility and inifinite horizon. Unlike in consumption CAPM, here returns are given and we are trying to figure out what does it mean for the optimal allocation. First consider **changing interest rates, constant risk premia and constant second and higher moments of returns**. The first order condition for the EZ utility function from equation (**??**) where we replace the wealth portoflio with portfolio of the agent as the relevant covariance term

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \gamma cov_t(r_{i,t+1}, r_{p,t+1}) + (\gamma-1)cov_t\left(r_{i,t+1}, (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{p,t+1+j}\right)$$

Under constant risk premium $r_{i,t+1} = r_{f,t+1} + rp$ all assets move in parallel with the risk free rate so that innovations in asset return are only due innovations in risk free rate

$$(E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{p,t+1+j} = (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{f,t+1+j}$$

Now plug this into the FOC from above

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \gamma cov_t(r_{i,t+1}, r_{p,t+1}) + (\gamma - 1)cov_t\left(r_{i,t+1}, (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{f,t+1+j}\right)$$

and consider a simple case with only 1 risky asset when covariance with the portfolio smplifies to

$$cov_t(r_{i,t+1}, r_{p,t+1}) = cov_t(r_{i,t+1}, \alpha_t r_{p,t+1}) = \alpha_t\sigma^2$$

Then the FOC becomes

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \gamma\alpha_t\sigma^2 + (\gamma - 1)cov_t\left(r_{i,t+1}, (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{f,t+1+j}\right)$$

Optimal risky asset allocation is then

$$\alpha_t = \underbrace{\frac{1}{\gamma}\frac{E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2}}{\sigma^2}}_{\text{Myopic Portfolio Demand}} + \underbrace{\left(1 - \frac{1}{\gamma}\right)\frac{1}{\sigma^2}cov_t\left(r_{i,t+1}, -(E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{f,t+1+j}\right)}_{\text{Interest Rate Hedging Term}}$$

Note that the two terms are weighted by $\frac{1}{\gamma}$ and $1 - \frac{1}{\gamma}$. As γ increases and the investor becomes more risk averse the second term dominates. COnsider the case of an infinitely risk-averse investor with $\gamma \to \infty$. In this case the optimal allocation is

$$\alpha_t = \frac{1}{\sigma^2}cov_t\left(r_{i,t+1}, -(E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{f,t+1+j}\right)$$

Inifnitely risk averse investor wants to invest in assets that perform well when future interest rates go down. Real perpetuities that go down in price when future interest rates go up offer such a product for this investor. Notice that in the 1 period horizon problem the optimal allocation of an infinitely risk averse investor was to invest zero in the risky asset and everything in the short term treasury. This is no longer the case for an infinitely lived agent as we just saw.

Application to the Asset Allocation Puzzle Financial advisors were recommending conservative investors to allocate higher share of their wealth into risky nominal bonds. This was seen as the violation of the Mutual Funds theorem that suggests the same risky asset allocations. However, from the perspective of the model that we just covered, bonds can offer a hedge against declines in interest rates and, therefore, may be preferred by conservative investors. Campbell and Viceira (2001) show that fr historical subsample where interest rates are persistent, the optimal portfolio for agents with high γ is indeed to invest a large fraction of wealth into bonds.

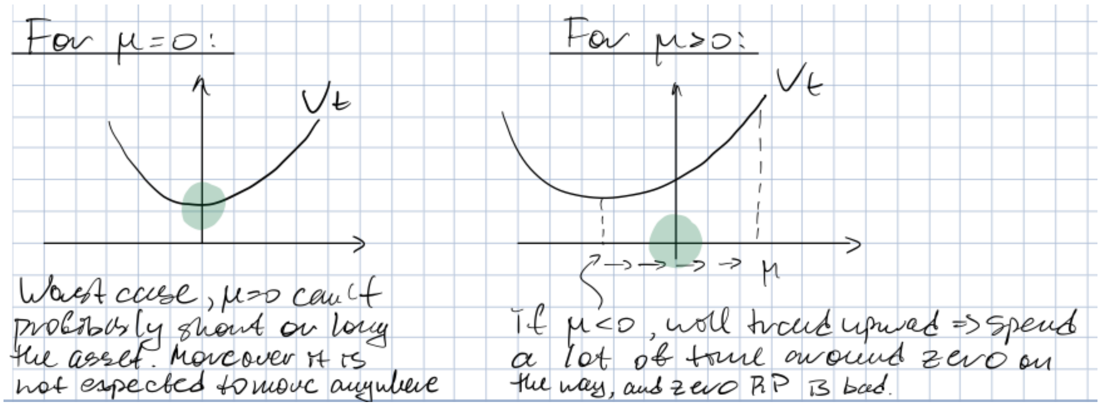
3.2 Hedging Risk Premia

Suppose now that the risk free rate is constant and there is a single risky asset with return r_{t+1} such that

$$\begin{aligned}
\text{Unexpected Change: } & r_{t+1} - E_t r_{t+1} = u_t \implies r_{t+1} = E_t r_{t+1} + u_{t+1} \\
\text{Risk Premium: } & E_t r_{t+1} - r_f + \frac{\sigma^2}{2} = x_t \\
\text{Process for RP: } & x_{t+1} = \mu + \phi(x_t - \mu) + \eta_{t+1} \\
& \sigma_{u\eta} < 0
\end{aligned}$$

So that the risk premium is mean reverting to μ. When $\sigma_{u\eta} < 0$ it means that when current return is high, risk premium moves down on average so that the future returns are low. Therefore, on average high returns are followed by low returns $\implies$ mean-reversion in returns.

We don't solve the model in the textbook, just assume that the risky asset share is affine in risk premium $\alpha_t = a_o + a_1 x_t$ and that the value function take the following form $V_t = \exp(b_0 + b_1 x_t + b_2 x_t^2)$. The coefficients b_0, b_1 and b_2 turn out to be in such a way so that the value function as a function for x_t looks like this



- When $\mu = 0$ the inconditional risk premium is equal to zero. It means that the worst place to be in terms of the value function is exactly when $x_t = 0$. This is because when $x_t = 0$ you can't profitably long or short. Moreover, you expect to stay at $x_t = 0$

- When $\mu > 0$ meaning that the unconditional risk premium is greater than zero, the worst place to be in terms of the value function is when $x_t < 0$. To see why consider the following experiment. Suppose that you are located in the minimum point of the value function on the right figure. You are expected to trend upward which means that you are going to spend a lot of time around $x_t = 0$. In $x_t \approx 0$ you can't profitably long or short the risk asset. However, if you start from $x_t = 0$ you trend upward and, therefore, will spend only a little time arond $x_t = 0$

Now we consider what are the slopes and the intercepts of the portfolio allocation rule $\alpha_t = a_0 + a_1 x_t$ as a function of risk premium.

- **Intercept** a_0**.** When $\mu = 0$ then $a_0 = 0$. Then when $x_t = 0$ agent holds zero share of the risky asset because he can't profitably long or short it and he is not expecting to go anywhere. However, it is not the case when $\mu > 0$ and $\sigma_{u\eta}$ – innovations to asset return and risk premium are negatively correlated. In fact an investor will have $a_0 > 0$ meaning that even when $x_t = 0$ he will hold positive share of the risky asset. This was impossible in the case of one periods investment. Why is it the case? Suppose that you are sitting at $x_t = 0$ (look at the right figure from above). As was discussed previously, it is bad for the agent to be in $x_t < 0$. Therfore, the agent wants to hedge the possibility that the risk premium go down. In the case when $\sigma_{u\eta} < 0$ the risky asset goes up whenever risk premium goes down. Therefore, he holds the risky asset even if $x_t = 0$ as **it provides a natural hedge against a declining risk premium**

- **Slope** a_1 Under $\gamma > 1$ and $\sigma_{u\eta} < 0$ the slope is higher than that of the myopic investor. Long-term investors time the market more aggressively than short term investors. This happens because stocks in this setting offer a hedge against a declining risk premium.

Empirical Estimation from a VAR model

- There is a mean reversion in in both bond and stock returns, but the risk of investing in treasury bills is increasing with horizon because of uncertainty about the rate a which it can be rolled over.

- Portfolio rule for for stocks features a hump shaped intertemporal hedging demand in risk tolerance $(1/\gamma)$. It first increases as $\gamma \uparrow$ and then goes down:

  - Log-utility investors don't have a intertemporal hedgind demand

– Infinitely risk averse agents don't invest in stocks anyway so that risk premium is not relevant for their decision and, hence, shouldn't be hedged.

- Conversely, share of bonds is U-shaped in risk aversion.

Hedging Volatility We can generalize this setting for time-varying volatility. For a log-normal setting investment opportuinities are summarized by the maximum available sharpe ratio. Hence, there will be a volatility hedging demand if the Sharpe ratio is going to move over time.

4 Intertempotal CAPM

Here we are going to appply intertemporal portfoloi choice theory to stidy the cross section of stock return. The question that we are asking is **what average stock returns or portfolios sorted on characteristics (such as value) would make a long term investor to hold the market portfolio rather than overweighting stocks with higher expected returns**. For a short term investor or long term investor with log utility the answer is CAPM: average excess stock returns over the risk free rate must be proportional to betas of stocks with respect to the market. Merton extended CAPM to multiple periods to ICAPM and howed that variables that predict future market returns are priced risk factors (I show this in the appendix).

First, let's use the CS return approximation for the market return

$$\tilde{r}_{m,t+1} = (E_{t+1} - E_t)\sum_{j=0}^{\infty}\rho^j \Delta d_{t+1+j} - (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{t+1+j} = N_{CF,t+1} - N_{DR,t+1}$$

For EZ investor the FOC from the previous section implies (remember that we are trying to make the investor to hold the market, hence, his portfolio is the market $\implies p \equiv m$):

$$\begin{aligned}
E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} &= \gamma cov_t(r_{i,t+1}, r_{m,t+1}) + (\gamma - 1)cov_t\left(r_{i,t+1}, (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{m,t+1+j}\right) \\
&= \gamma cov_t(r_{i,t+1}, N_{CF,t+1} - N_{DR,t+1}) + (\gamma - 1)cov_t\left(r_{i,t+1}, N_{DR,t+1}\right) \\
&= \gamma cov_t(r_{i,t+1}, N_{CF,t+1}) - cov_t\left(r_{i,t+1}, N_{DR,t+1}\right) \\
&= \gamma cov_t(r_{i,t+1}, N_{CF,t+1}) + cov_t\left(r_{i,t+1}, -N_{DR,t+1}\right)
\end{aligned}$$

Now define the following betas with respect to each component

$$\begin{aligned}
\beta_{i,CF,t} &= \frac{cov_t(r_{i,t+1}, N_{CF,t+1})}{\sigma_{mt}^2)} \\
\beta_{i,DR,t} &= \frac{cov_t(r_{i,t+1}, -N_{DR,t+1})}{\sigma_{mt}^2)} \\
\implies \beta_{imt} &= \beta_{i,CF,t} + \beta_{i,DR,t}
\end{aligned}$$

Hence, we can write the risk premium on asset i as

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \beta_{i,CF,t+1}\cdot\gamma\sigma_{mt}^2 + \beta_{i,DR,t+1}\cdot\sigma_{mt}^2 \tag{2}$$

The cash-flow beta has a risk price γ times higher than the price of risk of discount rate beta. This is because long-term ivestors fear permanent decline in wealth drive by cash-flows than they fear temporary declines in wealth due to higher discount rates. This is why cash-flow beta is called **bad beta** and discount rate beta is called **good beta**. There is logic about the relative prices cash flow vs. discount rate covariances in bonds. Suppose that we have a nominal bond that pays in two period. If you hold this bond to maturity you will for sure get 1 no matter how the interest rates will change the value of the bond in 1 period. However, cash flow news, for example, that the bond can default will influence both how much you can get in one period and in two periods.

Application to Growth Stocks Value anomaly shows that growth stocks have high betas but low returns. In light of the two beta model this means that this beta should come disproportionately from discount rate beta that has a low risk price. Therefore, growth stocks should do particularly well when future returns go down (risk premium goes down). Hence, returns of growth stocks should predict return on the market and there is some evidence of that. This allows the two-beta CAPM to explain the difference in returns for value and growth stocks with a relatively high $\gamma = 24$.

Some Caveats

- Empirical tests of this model consider only unconditional implications of this model, kind of considering an EZ investor holding a constant position in the market portfolio. However, this is only the case if the risk premium on the market stays constant
- This model can't explain the equity premium. As was discussed in chapter 5, most of the variation in the aggregate market comes from discount rate news meaning that DR beta is a dominant in market beta that is equal to 1: cash flow beta is only a small fraction of this unit beta. Hence, a large coefficient of RRA is needed to rationalize the equity premium.

4.1 Three-Beta Model

As was discussed briefly in chapter 6, volatility affects the quality of investment opportuinities. Therefore, it should affect intertemporal demand for a long-term investor. With time-varying volatility Campbell et al (2017) show that the innovation to the EZ SDF can be written as

$$\tilde{m}_{t+1} = -\gamma N_{CF,t+1} + N_{DT,t+1} + \frac{1}{2} N_{RISK,t+1}$$
$$\text{where } N_{RISK,t+1} = (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j Var_{t+j}(m_{t+1+j} + r_{t+1+j})$$

Thus $N_{RISK,t+1}$ is news about future risk. In order to substitute out future SDF in $N_{RISK,t+1}$, they make additional assumption that market returns and conditional variances follows an VAR(1). After all these manipulations they get that news about risk can be expressed as news about future variance of market returns

$$N_{RISK,t+1} = \omega(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \sigma_{m,t+j} = \omega N_{V,t+1}$$

When we put all the results together we get the following expression

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \gamma \sigma_{mt}^2 \beta_{CF,t} + \sigma_{mt}^2 \beta_{DR,t} - \frac{\omega \sigma_{mt}^2}{2} N_{i,V,t} \tag{3}$$

where ω is some function of γ so that γ is the only free parameter in this model.

Empirical Finding using VAR Estimation

- Find the growth stocks tend to outperform value stocks when long-term volatility forecasts increase (think about the technology boom of late 90s and the global financial crisis). $\Longrightarrow$ growth stocks have positive variance beta and since the risk of this covariance is negative in equation (3), they provide a hedge against deteriorating investment opportuinities due to higher volatility.
- This model fits the value premium and some other cross-sectional patterns in stock returns with a lower RRA of about 7.
- During the period studied, the aggregate stock market has a positive variance beta $\Longrightarrow$ need a large RRA to fit the equity premium since the aggregate stock market is a good hedge against increasing volatility $\Longrightarrow$ agents are willing to hold it even for large RRA.

5 Term Structure of Risky Assets

Duration-Style Explanation One potential explanation for high discount beta of growth stocks is that they have growing cash flows and, hence, derive a lot of their value from a distant future. Speaking in terminology from fixed income growth stocks have high duration. Recall that large duration means high sensitivity to movements in interest rates. Therefore, growth stocks are more sensitive to discount rate news than value stocks.

More Direct Analysis of Term Structure of Risk Premium Work of van Binsbergen, Brandt and Koijen (BBK 2012) construct a claim to dividends in the next 0.5 to 2 years and measures the properties of such short claim. They found that these claims offer high return and high risky but their Sharpe ratio is higher than that of the market. This suggests that short term claims may be more risky than long term claim $\implies$ downward sloping term structure of risk premium. Importantly they don't look at buy and hold claims on dividends (that are smooth since dividends are smooth) but rather at a strategy that holds these claim for one month and the rebalances.

Term Structure of Risk Premium in Baseline Models Downward sloping risk premium is hard to reconcile with standard consumption based models. Here we discuss several examples

- **Power Utility and iid Consumption Growth.** Consider a claim to time $t+n$ consumption (consumption strip). The return on this claim is

$$\frac{V_{n-1,t+1}}{V_{nt}} = \frac{E_{t+1}\left[\beta^{n-1}\left(\frac{C_{t+n}}{C_{t+1}}\right)^{-\gamma} C_{t+n}\right]}{E_t\left[\beta^{n}\left(\frac{C_{t+n}}{C_{t}}\right)^{-\gamma} C_{t+n}\right]} = \frac{E_{t+1}\left[\left(\frac{C_{t+n}}{C_{t+1}}\right)^{1-\gamma} C_{t+1}\right]}{\beta E_t\left[\left(\frac{C_{t+n}}{C_{t}}\right)^{1-\gamma} C_{t}\right]}$$

$$= \frac{C_{t+1}E_{t+1}\left[\left(\frac{C_{t+n}}{C_{t+1}}\right)^{1-\gamma}\right]}{C_t\beta E_t\left[\left(\frac{C_{t+n}}{C_{t}}\right)^{1-\gamma}\right]} = \frac{C_{t+1}E_{t+1}\left[\left(\frac{C_{t+2}}{C_{t+1}}\right)^{1-\gamma}\left(\frac{C_{t+3}}{C_{t+2}}\right)^{1-\gamma}\cdots\right]}{C_t\beta E_t\left[\left(\frac{C_{t+1}}{C_{t}}\right)^{1-\gamma}\left(\frac{C_{t+2}}{C_{t+1}}\right)^{1-\gamma}\cdots\right]} = \frac{C_{t+1}/C_t}{\beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]}$$

  where we were able to cancel all the terms since consumption growth is iid and, therefore:

$$E_{t+1}\left[\left(\frac{C_{t+j+1}}{C_{t+j}}\right)^{1-\gamma}\right] = E_t\left[\left(\frac{C_{t+j+1}}{C_{t+j}}\right)^{1-\gamma}\right] \text{ for } j \geq 1$$

  Return to consumption strips $\frac{V_{n-1,t+1}}{V_{nt}}$ is independent of n so that the term structure of risk premium is flat

- **Long-Run Risks Models** There are two components to consider

  1. Shocks that decrease future consumption growth increase SDF $\implies$ raise interest rates $\implies$ drive bond prices up. Since consumption is dividend this also drives dividends down $\implies$ stock prices fall
  2. Shocks that increase volaility of consumption growth increase SDF $\implies$ drive bond prices up. These shocks increase precautionary savings motives $\implies$ stock prices decrease

  Both of these effects are more pronounced at longer horizons.

- **Habit Model of Campbell Cochrane.** In this model shocks that drive surplus down increase marginal utility not only today but also in the future since consumption is expected to stay close to habit for a long time. This makes longer maturity claim riskier.

Response from Lettau and Wachter (2007,2011) They propose the following model

$$\Delta d_{t+1} = z_t + \sigma + d\varepsilon_{d,t+1}$$
$$z_{t+1} = (1-\phi_z)\bar{z} + \phi_z z_t + \sigma_z \varepsilon_{z,t+1}$$
$$x_{t+1} = (1-\phi_x)\bar{x} + \phi_x x_t + \sigma_x \varepsilon_{x,t+1}$$
$$m_{t+1} = -r_f - \frac{x_t^2}{2} - x_t \varepsilon_{d,t+1}$$

that feature a reduced form essentially affine model for SDF that is modified to price risky assets. Notice that the only shock that enter SDF is $\varepsilon_{d,t+1}$ – it is the only shock that is directly priced. Hence, by itself the decrease in growth rate of dividends doesn't affect marginal utility (as is the case in long-run risk model where shocks to future consumption directly affect marginal utility). They calibrate this model to have a negative correlation between innovations to dividend growth and contemporaneous dividends: $\varepsilon_{z,t+1}$ and $\varepsilon_{d,t+1}$. Thus, dividend exhibit mean-reversion $\Longrightarrow$ long-term dividends are less risky than short term dividends.

6 Learning

Main Intuition When investor is uncertain about the process that governs returns (for example, he doesn't know the mean return) there is positive covariance between the realized return and expected future returns. This happens because a positive return, for example, updates the mean return upwards. Alternatively, negative return updates mean return downwards. This creates larger long-run volatility and generates negative intertemporal hedging demand for the investor. The main problem that this can't persists forever since over time the agent's estimate of the mean will be more and more precise so that each realization of return will not shift investor's belief that much. Hence, intertemporal hedging demand will decline in absolute value over time.

Deriving the Model Suppose that return is given by y_t and it follows

$$y_{t+1} = \mu + \sigma \varepsilon_{t+1}$$

where μ is unknown and is being learnt over time. First, introduce some notataion

- Prior for μ is $p_0(\mu) = \mathcal{N}(\mu_0, A_0\sigma^2)$ where we scale the variance by σ^2 just for analytical convenience.
- After an agent observes a stream of return $y_1, \ldots, y_t$ his posterior is

$$p_t(\mu) = p(\mu|y_1, \ldots, y_t) = \mathcal{N}(\mu_t, A_t\sigma^2)$$

- When the investor forms his expectations about $t+1$ he doesn't know the true mean. Therefore, both uncertainty about μ and the random noise ε_{t+1} will enter into his decision. The one period variance is

$$var_t(y_{t+1}) = \underbrace{A_t\sigma^2}_{\text{uncertainty about return mean}} + \underbrace{\sigma^2}_{\text{noise in returns}}$$

  and the mean is simple

$$E_t[y_{y+1}] = \mu_t$$

  Since y_{t+1} is normal, we can summarize *equivalently* say that the agent view the next period return as

$$y_{t+1} = \mu_t + \sqrt{A_t + 1}\sigma\tilde{\varepsilon}_{t+1}$$

  Note that this modified y_{t+1} has the same conditional mean and variance as the true y_{t+1} derived above.

Now consider how the updating works with the notation that we described. Recall that in the normal bayesian setting the posterior mean is the precision weighted average of the prior and the signal. The precision of the prior is $(A_t\sigma^2)^{-1}$ and the precision of the signal is always σ^{-2}. Hence, the posterior mean after observing signal y_{t+1} is

$$\mu_{t+1} = \frac{\frac{1}{A_t\sigma^2}\mu_t + \frac{1}{\sigma^2}y_{t+1}}{\frac{1}{A_t\sigma^2} + \frac{1}{\sigma^2}} = \frac{\mu_t + A_t y_{t+1}}{1 + A_t}$$

using the equivalent notation with $\widetilde{\varepsilon}_{t+1}$ we have

$$\mu_{t+1} = \frac{\mu_t + A_t(\mu_t + \sqrt{A_t+1}\sigma\widetilde{\varepsilon}_{t+1})}{1+A_t} = \mu_t + \frac{A_t\sigma}{\sqrt{A_t+1}}\widetilde{\varepsilon}_{t+1}$$

The posterior precision is

$$\frac{1}{A_{t+1}\sigma^2} = \frac{1}{A_t\sigma^2} + \frac{1}{\sigma^2} \implies \frac{1}{A_{t+1}} = \frac{1}{A_t} + 1 \implies A_{t+1} = \left(\frac{1}{A_t} + 1\right)^{-1}$$

Implications for Long Term Variance First, consider how learning affect the variance of a long term K period return from the prespective of the agent. Denote variance conditional on agent's information set as $Var_t^*(\cdot)$. Then[1]

$$\begin{aligned} Var_t^*(y_{t+1} + \cdots + y_{t+K}) &= E_t^*[Var(y_{t+1} + \cdots + y_{t+K}|\mu)] + Var_t^*(E[y_{t+1} + \cdots + y_{t+K}|\mu]) \qquad \text{[Use } y_{t+j}|\mu \sim iid\text{]} \\ &= E_t^*[\sigma^2 K] + Var_t^*(K\mu) \\ &= \sigma^2 K + K^2 Var_t^*(\mu) \\ &= \sigma^2 K + K^2 A_t \sigma^2 \\ &= \sigma^2 K(1 + +KA_t\sigma) \implies \frac{Var_t^*(y_{t+1} + \cdots + y_{t+K})}{K} \text{ increases in } K \end{aligned}$$

Unlike in the model with *iid* return, uncertainty about the parameter values increases the long-run annualized variance. This means that from the perspective of the agent risk increases with horizon and we will see shortly that this creates intertemporal hedging motives

Learning and Intertemporal Hedging For the intertemporal hedging we need to calculate the covariance between the return and revisions in future return, again, from the perspective of the agent

$$\begin{aligned} &Cov_t^*(y_{t+1}, (E_{t+1}^* - E_t^*)\sum_{j=1}^{\infty}\rho^j y_{t+1+j}) \\ &= Cov_t^*(\mu_t + \sqrt{1+A_t}\sigma\tilde{\varepsilon}_{t+1}, (E_{t+1}^* - E_t^*)\sum_{j=1}^{\infty}\rho^j \mu_{t+j} + \sqrt{1+A_{t+j}}\sigma\tilde{\varepsilon}_{t+1+j}) \\ &= Cov_t^*(\sqrt{1+A_t}\sigma\tilde{\varepsilon}_{t+1}, (E_{t+1}^* - E_t^*)\sum_{j=1}^{\infty}\rho^j \mu_{t+j}) \\ &= Cov_t^*(\sqrt{1+A_t}\sigma\tilde{\varepsilon}_{t+1}, (E_{t+1}^* - E_t^*)\left[\rho(\mu_t + \frac{\sigma A_t}{\sqrt{1+A_t}}\tilde{\varepsilon}_{t+1}) + \rho^2(\mu_t + \frac{\sigma A_t}{\sqrt{1+A_t}}\tilde{\varepsilon}_{t+1} + \frac{\sigma A_t}{\sqrt{1+A_t}}\tilde{\varepsilon}_{t+2}) + \dots\right]) \\ &= Cov_t^*(\sqrt{1+A_t}\sigma\tilde{\varepsilon}_{t+1}, \left[\rho\frac{\sigma A_t}{\sqrt{1+A_t}}\tilde{\varepsilon}_{t+1} + \rho^2\frac{\sigma A_t}{\sqrt{1+A_t}}\tilde{\varepsilon}_{t+1} + \dots\right]) \\ &= Cov_t^*(\sqrt{1+A_t}\sigma\tilde{\varepsilon}_{t+1}, \frac{\rho}{1-\rho}\frac{\sigma A_t}{\sqrt{1+A_t}}\varepsilon_{t+1}) = \frac{\rho}{1-\rho}\sigma^2 A_t > 0 \end{aligned}$$

This means that the agent has negative intertemporal hedging if his risk aversion $\gamma > 1$. However, the strength of this effect goes to zero as time passes since A_t goes down as more signals are received.

[1]The law of total variance says $Var(Y|X_1) = E[Var(Y|X_1, X_2)|X_1] + Var(E[Y|X_1, X_2]|X_1)$ and we assume that X_1 is the information set of the investor and X_2 is the true μ. Hence, (X_1, X_2) is the infromation set of the investor augmented with the knowledge of the true μ.