Asset Pricing Notes. Chapter 8: Fixed Income Securities

Roman Sigalov

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1 Introduction and Definitions

Basic Concepts Price of a Zero Coupon Bond (ZCB) that pya s1 at time $t+n$ is P_{nt}. Yield-to-Maturity (YTM) is defined as

$$Y_{nt} : P_{nt} = \frac{1}{(1+Y_{nt})^n} \implies p_{nt} = -ny_{nt} \implies y_{nt} = -\frac{1}{n}p_{nt}$$

Since bonds are not inifniely lived their price is not "stationary" meaning that we need to keep track of time to maturity when calculating returns. In particular, the holding period return of a ZCB is

$$1 + R_{n,t+1} = \frac{P_{n-1,t+1}}{P_{nt}} = \frac{(1+Y_{nt})^{-n}}{(1+Y_{n-1,t+1})^{n-1}}$$

there are not intermediate payments and return comes solely from capital gains/losses. In logs

$$r_{n,t+1} = p_{n-1,t+1} - p_{nt} = -(n-1)y_{n-1,t+1} + ny_{nt} = \underbrace{y_{nt}}_{\text{initial yield}} + \underbrace{(n-1)(y_{nt} - y_{n-1,t+1})}_{\text{change in yield}}$$

To get a high return you need to have a bond with (1) high initial yield y_{nt} and (2) declining yield $(y_{nt} - y_{n-1,t+1})$. Note, however, that declining yield component compares bonds with different maturity. To make the quantities comparable add and subtract $(n-1)y_{n,t+1}$

$$\begin{aligned} r_{n,t+1} &= y_{nt} + (n-1)(y_{nt} - y_{n-1,t+1}) + (n-1)y_{n,t+1} - (n-1)y_{n,t+1} \\ &= y_{nt} + \underbrace{(n-1)(y_{nt} - y_{n,t+1})}_{\Delta \text{ in yield curve}} + \underbrace{(n-1)(y_{n,t+1} - y_{n-1,t+1})}_{\text{"riding the yield curve"}} \end{aligned}$$

Thus we get the change in the constant maturity bond which is small empirically and doesn't contribute to return a lot. Second, we have a "riding the yield curve" component. This component depends on the slope of the yield curve that is positive meaning that the yield curve is upward sloping. Excess return is

$$\begin{aligned} r_{n,t+1} - r_{f,t+1} &= r_{n,t+1} - y_{1t} \\ &= \underbrace{y_{nt} - y_{1t}}_{\text{term spread}} + (n-1)(y_{n,t} - y_{n-1,t+1}) \\ &= s_{nt} + (n-1)(y_{n,t} - y_{n-1,t+1}) \end{aligned}$$

Some Empirical Facts about Components

1. $y_{n-1,t+1} - y_{nt}$ is typically negative and is decreasing with in absolute value with maturity (yield curve is generally concave in maturity). However, its effect is multiplied by $(n-1)$ so that small variations in yield can significantly affect the price of long bonds.

2. Constant maturity spread $y_{n,t+1} - y_{nt}$ is declining over time: we have lower interest rates now, so that the whole yield curve moves down.

3. Excess return $r_{n,t+1} - y_{1t}$ is positive and increases with maturity $\implies$ **term premium**

Forward Rates Forward rate F_{nt} is the rate determined at time t to lend or borrow for one period at rate F_{nt} at time $t+n$. We can construct forward rate using a law of one price argument. At time t do the following

1. Buy a bond with maturity $n+1$. Thus at time t get outflow of $P_{n+1,t}$ and at time $t+n+1$ get 1

2. Now we need to zero this initial outflow and create an outflow at time $t+n$. We can finance buying this bond at time t by selling short $\frac{P_{n+1,t}}{P_{nt}}$ of n period bonds.

Now look at the cash flows that we have from these trades

Trade	t	$\cdots$	$t+n$	$t+n+1$
$(n+1)$-bond	$-P_{n+1,t}$	$\cdots$	0	$+1$
n-bond	$+\frac{P_{n+1,t}}{P_{nt}} P_{nt}$	$\cdots$	$-\frac{P_{n+1,t}}{P_{nt}}$	$+1$
Total	0	$\cdots$	$-\frac{P_{n+1,t}}{P_{nt}}$	1

Thus with this trade we can guarantee ourselves zero flows today and ability to borrow at at period $t+n$ for one period. This is exactly what forward conract is doing. Therefore, the outflow at time $t+n$ should be the price of forward bond meaning that the forward rate should be an inverse of it

$$\begin{aligned} 1 + F_{nt} = \left(\frac{P_{n+1,t}}{P_{nt}}\right)^{-1} \implies f_{nt} &= p_{nt} - p_{n+1,t} \\ &= -ny_{nt} + (n+1)y_{n+1,t} \\ &= y_{nt} - (n+1)y_{nt} + (n+1)y_{n+1,t} \\ &= y_{nt} + (n+1)(y_{n+1,t} - y_{nt}) \end{aligned}$$

Forward rate is larger than the bond rate ($f_{nt} > y_{nt}$) if the yield curve is upward sloping ($y_{n+1,t} - y_{nt} > 0$). We can think about yield on a bond as the average rate at which we can borrow for n periods. Then forward rate is the marginal rate at which we can extend the borrowing for one more period. Hence, forward and yield curve resemble marginal and average costs and we know that for average and marginal costs marginal cost curve lies above the average cost curve when it's rising.

We can express return using forward rates in the following way

$$\begin{aligned} r_{n,t+1} &= p_{n-1,t+1} - p_{nt} \\ &= (p_{n-1,t+1} - p_{n,t+1}) + (p_{n,t+1} - p_{nt}) \\ &= f_{n-1,t+1} - n\Delta y_{n,t+1} \end{aligned}$$

if the yield curve is not trending then we have that unconditionally

$$Er_{n,t+1} = Ef_{n-1,t+1}$$

The implication of this is that if average simple return is constant with maturity

$$Er_{n,t+1} + \frac{\sigma_n^2}{2} = const \implies Er_{n,t+1} = const - \frac{\sigma_n^2}{2}$$

and σ_n^2 increases with maturity **as is the case in the data**, then forward rates are falling with maturity.

2 Expectation Hypothesis of The Term Structure

Expectation Hypothesis (EH) says that expected excess return on long-term bonds is constant over time, i.e. there is no particularly bad or good time to prefer (and buy) long bonds over short bonds. In the pure expectations hypothesis this constant is zero. There is a question of how to formulate EH: in average return or in logs. We will show why formulating it in average returns raises problems. Suppose that Pure EH holds for two different maturities

1. For one period

$$\underbrace{1 + Y_{1t}}_{\text{Hold 1 period bonds}} = \underbrace{E_t[1 + R_{2,t+1}]}_{\text{Hold 2-period bond for 1 period}}$$

Use the definition of return to substitute

$$1 + R_{2,t+1} = \frac{P_{1,t+1}}{P_{2t}} = \frac{(1 + Y_{2,t})^2}{1 + Y_{1,t+1}}$$

$$\implies 1 + Y_{1t} = E_t\left[\frac{(1 + Y_{2,t})^2}{1 + Y_{1,t+1}}\right] = (1 + Y_{2,t})^2 E_t\left[\frac{1}{1 + Y_{1,t+1}}\right]$$

2. For two periods

$$\underbrace{(1 + Y_{2t})^2}_{\text{Holding 2-period bond for 2 period}} = \underbrace{(1 + Y_{1t})E_t[1 + Y_{1,t+1}]}_{\text{Holding 1-period bonds and rolling over}}$$

this implies

$$1 + Y_{1t} = (1 + Y_{2t})^2 \frac{1}{E_t[1 + Y_{1,t+1}]}$$

Because of Jensen's inequality we have that

$$(1 + Y_{2t})^2 \frac{1}{E_t[1 + Y_{1,t+1}]} \neq (1 + Y_{2,t})^2 E_t\left[\frac{1}{1 + Y_{1,t+1}}\right]$$

unless $Y_{1,t+1}$ is deterministic.

Because of the contradiction outlined above it is preferable to work with EH written in logs:

$$E_t[r_{n,t+1} - y_{1t}] = \mu_n \neq \text{function of time} \tag{1}$$

In pure expectation hypothesis $\mu_n = 0$. Alternatively, we can write EH in the following form

$$E_t\left[y_{nt} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] = \theta_n \neq \text{function of time} \tag{2}$$

where the first term is yield of holding n period bond to maturity, and the second term is a sequence of return of rolling over 1 period bond. We can move from one representation to another. To see how, first notice that cumulated returns of holding an n period bond to maturity is $n \cdot y_{nt}$

$$\begin{aligned} r_{n,t+1} + r_{n-1,t+2} + \cdots + r_{1,t+n} &= (p_{n-1,t+1} - p_{nt}) + (p_{n-2,t+2} - p_{n-1,t+1}) + \cdots + (\log(1) - p_{1,t+n-1}) \\ &= -p_{nt} = n \cdot y_{nt} \end{aligned}$$

Using this replace y_{nt} with the the average return in equation (2).

$$
\begin{aligned}
E_t\left[y_{nt} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] &= E_t\left[\frac{1}{n}\left(r_{n,t+1} + r_{n-1,t+2} + \cdots + r_{1,t+n}\right) - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] \\
&= E_t\left[\frac{1}{n}\sum_{i=0}^{n-1} r_{n-i,t+1+i} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] \\
&= E_t\left[\frac{1}{n}\left(\sum_{i=0}^{n-1} r_{n-i,t+1+i} - y_{1,t+i}\right)\right] \\
&= E_t\left[\frac{1}{n}\left(\sum_{i=0}^{n-1} \mu_{n-i}\right)\right] \neq \text{function of time}
\end{aligned}
$$

where in the last equality we used the form of expectation hypothesis in equation (1).

2.1 Restrictions on Interest Rate Dynamics

Restriction 1 Now we consider what are the restrictions that EH poses on interest rate dynamics. First, use the expression for return $r_{n,t+1} = y_{nt} - (n-1)(y_{n-1,t+1} - y_{nt})$ and plug it into equation (1):

$$
\begin{aligned}
E_t[y_{nt} - (n-1)(y_{n-1,t+1} - y_{nt}) - y_{1t}] &= \mu_n \\
E_t[s_{nt} - (n-1)(y_{n-1,t+1} - y_{nt})] &= \mu_n \\
\implies s_{nt} = \mu_n + E_t[(n-1)(y_{n-1,t+1} - y_{nt})] \qquad (3)
\end{aligned}
$$

This equation says that when term spread is too large, then future yields are expected to increase. This **generate capital losses to offset high initial yield of the bond y_{nt}**. At first this seems counterintuitive as the increase in long yields should increase the spread further. The next restriction shows why is this the case

We can test this prediction with the following regression

$$
(n-1)(y_{n-1,t+1} - y_{nt}) = \alpha + \beta s_{nt} + \varepsilon_{t+1}
$$

where the null is the $\beta = 1$ and α is unrestricted. This

Restriction 2 Next, rearrange equation (2) to get

$$
\begin{aligned}
E_t\left[y_{nt} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] &= \theta_n \\
E_t\left[y_{nt} - y_{1t} + y_{1t} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] &= \theta_n \\
E_t\left[s_{nt} + \left(\frac{1}{n} + \frac{n-1}{n}\right) y_{1t} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] &= \theta_n
\end{aligned}
$$

Using this we can derive the following:

$$
s_{nt} = E_t \sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) \Delta y_{1,t+i} \qquad (4)
$$

This means that when spread is high, the future short term rates are expected to increase. When we consider the previous puzzling fact that when spread is large then long term rate is expected to increase, prediction 2 suggests that short rates also going up. In fact, short rates are expected to go up more so that the spread is actually expected to shrink.

Takes together predictions 1 and 2 imply that when spread is high

1. Long term rates are expected to go up
2. Short term rates are expected to go up
3. Short rates go up more than long rate so that the spread is expected to shrink.

We can test prediction 2 with the following regression

$$\sum_{i=0}^{n-1}\left(1-\frac{i}{n}\right)\Delta y_{1,t+i}=\alpha+\beta s_{nt}+\varepsilon$$

Restriction 3 The last restriction is about forward rates. n period forward rate is

$$f_{nt}=p_{nt}-p_{n+1,t}=-ny_{nt}+(n+1)y_{n+1,t}$$

Use EH in equation (2) to substitute for y_{nt} and $y_{n+1,t}$:

$$ny_{nt}=n\theta_n+E_t\sum_{i=0}^{n-1}y_{1,t+i}$$

Then the forward rate is

$$\begin{aligned}f_{nt}&=-ny_{nt}+(n+1)y_{n+1,t}\\&=-\left(n\theta_n+E_t\sum_{i=0}^{n-1}y_{1,t+i}\right)+\left((n+1)\theta_{n+1}+E_t\sum_{i=0}^{n}y_{1,t+i}\right)\\&=[-n\theta_n+(n+1)\theta_{n+1}]+E_ty_{1,t+n}\end{aligned}$$

$$f_{nt}=\phi_n+E_ty_{1,t+n} \tag{5}$$

EH implies that the variation in the forward rate comes from the variation in the expectation about future short rates. Hence, forward rates can be used to elicit market expectations about the future short term rates. In pure EH we have ϕ_n so that forward rate is exactly the expectation of future short rate

Empirical Tests

- Not surprisingly, the empirical tests tend to reject the implications of EH. However, they do it in a different way. When we test restriction 1 in equation (3) we tend to have a negative coefficients on different maturities larger than 1 in absolute value. Remember that the null under EH is $\beta=1$. This means that large spread today predicts decline in the long yield in the future
- When we test restriction 2 in equation (4) we tend to get positive coefficients but which are insignificant and small in magnitude compared to the null hypothesis under EH that $\beta=1$
- Campbell and Shiller (1991) argue that this this behavior is consistent with variation in the long yield not related to changes in the expectations of future short term rates. For example, it can be the that term premium is **temporary** changing over time. This will move around s_{nt} in equation (4) but will leave expectation of future short rates unchanged. Thus, this introduces variation on the right hand side without any effect on the left hand side. This **temporary** variation also moves left and right hand side of equation (3) in opposite directions thus making the coefficient negative.
- Cochrane and Piazzesi (2005) find that some combination of forward rates predicts excess returns on bonds going forward. However, this result is not that strong in bigger samples

3 Affine Term Structure Models

Consider the price of an n period zero coupon bond

$$P_{nt} = E_t[M_{t+1}P_{n-1,t+1}] = E_t[M_{t+1}E_{t+1}[M_{t+2}P_{m-2,t+2}]] = \cdots = E_t[M_{t+1}\dots M_{t+n}\cdot 1] = E_t[M_{t,t+n}]$$

Hence, price of the bond is the expectation of the product of future SDFs. Thus, modelling the term structure of yields is essentially modelling the the term structure of SDF.

In this chapter we work with a reduce form SDF that is not derived from a particular utility function. This has the benefit that it rules out arbitrage. As usual we will assume that everything is jointly lognormal so that we can use the usual

$$\begin{aligned} p_{nt} &= E_t[m_{t+1} + p_{n-1,t+1}] + \frac{1}{2}Var_t(m_{t+1} + p_{n-1,t+1}) \\ &= \underbrace{\left[E_t m_{t+1} + \frac{1}{2}Var_t(m_{t+1}\right]}_{-r_{f,t+1}=-y_{1t}=p_{1t}} + E_t p_{n-1,t+1} + \frac{1}{2}Var_t(p_{n-1,t+1}) + cov_t(m_{t+1}, p_{n-1,t+1}) \\ &= p_{1t} + E_t p_{n-1,t+1} + \frac{1}{2}Var_t(p_{n-1,t+1}) + cov_t(m_{t+1}, p_{n-1,t+1}) \end{aligned} \tag{6}$$

Affine Term Structure Models will ensure that all prices of bonds are affine in state variables. This will be done by assuming a particular dependence of SDF on state variables and particular law of motion for state variables.

3.1 Completely Affine Term Structure Models

There is a single state variable x_t that evolves according to

$$x_{t+1} = \mu + \phi x_t + \sigma\varepsilon_{t+1} \tag{7}$$

Log-SDF is

$$m_{t+1} = -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 - \frac{\lambda}{\sigma}\varepsilon_{t+1} \tag{8}$$

The second term in the SDF is the Jensen's adjustment that will become clear below. Consider the price of a 1 period bond

$$\begin{aligned} p_{1t} &= E_t m_{t+1} + \frac{1}{2}Var_t(m_{t+1}) \\ &= -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 + \frac{1}{2}Var_t\left(\frac{\lambda}{\sigma}\varepsilon_{t+1}\right) \\ &= -x_t \end{aligned}$$

Then the risk free rate is

$$y_{1t} = -p_{1t} = x_t$$

so that the state variable x_t is exactly the short rate. Due to linearity of the model we can guess an affine solution in the state x_t of the price of an n-period bond

$$p_{nt} = A_n + B_n x_t$$

Now substitute this guessed form into the bond pricing equation (6)

$$
\begin{aligned}
A_n + B_n x_t &= -x_t + E_t[A_{n-1} + B_{n-1}x_{t+1}] + \frac{1}{2}Var_t(A_{n-1} + B_{n-1}x_{t+1}) + cov_t(A_{n-1} + B_{n-1}x_{t+1}, m_{t+1}) \\
&= -x_t + A_{n-1} + B_{n-1}E_t x_{t+1} + \frac{1}{2}B_{n-1}^2 Var_t(x_{t+1}) + cov_t(B_{n-1}x_{t+1}, -\frac{\lambda}{\sigma}\varepsilon_{t+1}) \\
&= -x_t + A_{n-1} + B_{n-1}(\mu + \phi x_t) + \frac{1}{2}B_{n-1}^2\sigma^2 + B_{n-1}cov_t(\sigma\varepsilon_{t+1}, -\frac{\lambda}{\sigma}\varepsilon_{t+1}) \\
&= -x_t + A_{n-1} + B_{n-1}(\mu + \phi x_t) + \frac{1}{2}B_{n-1}^2\sigma^2 - \lambda B_{n-1} \\
&= \underbrace{(A_{n-1} + B_{n-1}\mu + \frac{1}{2}B_{n-1}^2\sigma^2 - \lambda B_{n-1})}_{A_n} + \underbrace{(B_{n-1}\phi - 1)}_{B_n} x_t
\end{aligned}
$$

Now we need to match coefficients. First note that $B_1 = -1$ since $p_{1t} = -1 \cdot x_t$. Then using $B_n = B_{n-1}\phi - 1$

$$
\begin{aligned}
B_1 &= -1 \\
B_2 &= -1 - \phi \\
B_3 &= -1 - \phi - \phi^2 \\
B_n &= -1 - \phi - \phi^2 - \cdots - \phi^{n-1} = -\frac{1 - \phi^n}{1 - \phi} < 0
\end{aligned}
$$

Now we can match the constants to get

$$
A_n = A_{n-1} + B_{n-1}(\mu - \lambda) + \frac{1}{2}B_{n-1}^2\sigma^2
$$

that we can solve from the initial condition $A_1 = 0$ since $p_{1t} = -x_t$. Notice that λ which is the price of ε-risk only shows up modifying the drift of the process, i.e. in $(\mu - \lambda)$. **Bond prices in a model with risk are the same as in the model without risk but with a lower drift** $\mu \to \mu - \lambda$.

Risk Premium Now let's derive risk premium on the bond. Using the pricing equation for joint lognormal SDF and returns we have

$$
\begin{aligned}
r_{n,t+1} - y_{1t} + \frac{1}{2}Var_t(r_{n,t+1}) &= -cov_t(m_{t+1}, p_{n-1,t+1}) \\
&= -cov_t(-\frac{\lambda}{\sigma}\varepsilon_{t+1}, B_{n-1}x_{t+1}) \\
&= -cov_t(-\frac{\lambda}{\sigma}\varepsilon_{t+1}, B_{n-1}(\mu + \phi x_t + \sigma\varepsilon_{t+1})) \\
&= -cov_t(-\frac{\lambda}{\sigma}\varepsilon_{t+1}, B_{n-1}\sigma\varepsilon_{t+1}) \\
&= \lambda B_{n-1}
\end{aligned}
$$

- Decrease in ε_{t+1} drive bond prices up (since $B_n < 0$ in $p_{n,t+1} = A_n + B_n x_{t+1}$). At the same time, this shock drive SDF up when $\lambda > 0$. Hence, bond are doing good in bad times (as measure by higher SDF) and, therefore, they are a hedge. Hence, they have a negative risk premium $\lambda B_{n-1} < 0$ since $B_{n-1} < 0$.
- When $\lambda < 0$ then the reverse happens $\implies$ bonds go down when SDF goes up and risk premium is positive.

In this case the risk premium is constant and, therefore, the expectation hypothesis holds. However, the pure EH doesn't hold since the risk premium is positive.

Forward Rate In this model the forward rate is

$$\begin{aligned} f_{nt} &= p_{nt} - p_{n+1,t} \\ &= -(A_{n+1} - A_n) - (B_{n+1} - B_n)x_t \\ &= -B_n(\mu - \lambda) - \frac{1}{2}B_{n+1}^2\sigma^2 - \left(-\frac{1-\phi^n}{1-\phi} + \frac{1-\phi^{n+1}}{1-\phi}\right)x_t \\ &= \frac{1-\phi^n}{1-\phi}(\mu - \lambda) - \frac{1}{2}\left(\frac{1-\phi^n}{1-\phi}\right)^2\sigma^2 - \frac{\phi^{n+1}-\phi^n}{1-\phi}x_t \\ &= \underbrace{\frac{\mu-\lambda}{1-\phi}}_{\text{risk-adjusted average short rate}} - \frac{1}{2}\left(\frac{1-\phi^n}{1-\phi}\right)^2\sigma^2 + \phi^n \underbrace{\left[x_t - \frac{\mu-\lambda}{1-\phi}\right]}_{\text{deviation from r.a. average short rate}} \end{aligned}$$

The limiting forward rate (as $n \to \infty$) is

$$f_{\infty,t} = \frac{\mu-\lambda}{1-\phi} - \frac{1}{2}\left(\frac{1-\phi^n}{1-\phi}\right)^2\sigma^2$$

this is constant or diverges to infinity if $\phi \to \infty$. However, in the data forward rates move a lot and they don't diverge to infinity.

3.2 Completely Affine Heteroskedastic Models

The main drawback of the previous model was constant risk premium. In this model we are going to get time varying risk premium and still an affine price of bonds in the state variable. We assume the following law of motion

$$x_{t+1} = \mu + \phi x_t + \sigma x_t^{-1/2}\varepsilon_{t+1} \tag{9}$$

In continous time limit this process has the property that it doesn't go below zero as the volatility also goes to zero as $x_t \to 0$. We also change SDF equation to

$$m_{t+1} = -x_t + \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t - \frac{\lambda}{\sigma}x_t^{1/2}\varepsilon_{t+1} \tag{10}$$

We can apply the same approach of matching coefficients as before to derive expressions for B_n and A_n in $p_{nt} = A_n + B_n x_t$. However, the recursion is no longer linear an we can't solve for B_n in closed form as before

$$B_n = -1 + (\phi - \lambda)B_{n-1} + \frac{B_{n-1}^2}{2}$$

Now risk adjustment also enters the recursion for B_n and enters in the same way as persistence of the state process x_t.

Risk Premium The risk premium can be obtained in the same way as in the previous model

$$\begin{aligned} r_{n,t+1} - y_{1t} + \frac{1}{2}Var_t(r_{n,t+1}) &= -cov_t(m_{t+1}, p_{n-1,t+1}) \\ &= -cov_t(-\frac{\lambda}{\sigma}x_t^{-1/2}\varepsilon_{t+1}, B_{n-1}\sigma x_t^{-1/2}\varepsilon_{t+1}) \\ &= \lambda B_{n-1}x_t \end{aligned}$$

Now risk premium is time varying and it is determined by the state x_t. Since, x_t is exactly the the risk free rate, risk premium is proportional to the risk free rate. But we are still unhappy with these results

1. This model predicts that low yield spread $s_{nt} = y_{nt} - y_{1t} = y_{nt} - x_t$ i.e. higher x_t implies large risk premium and, therefore, low yield spread forecasts large return. However, the tests of EH suggest that when spread is low $\implies$ future long yields go up $\implies$ bond prices god down

2. Assumption that the interest rate volatility is proportional to the square root of the short rate doesn't fit the data well. Data from 60s to 80s suggests that volatility increases with a higher power than a square root. On the other hand, more recent data suggests that with lower rates and still high volatility the relationship has a lower power.

Additionally, we have that the variance of bond yields is proportional to the state variable x_t and, hence, to the risk premiu

3.3 Essentially Affine Term Structure Models

Completely affine term structure model with time varying risk premium implies that variance of bond yields is proportional to risk premium

$$var_t(y_{n,t+1}) = var_t(p_{n,t+1}) = var_t(B_n x_{t+1}) = var_t(B_n \sigma x_t^{1/2}) = B_n \sigma x_t$$

Duffee (2002) shows that it is possible to write an affine model in which variance varies indepedently of risk premium. In this way we don get get linearity of mean and variance of log SDF in the state variable, but bonds prices and yields are still linear in state. We have a homoskedsastic law of motion for the state variable

$$x_{t+1} = \mu + \phi x_t + \sigma \varepsilon_{t+1}$$

but log SDF becomes

$$m_{t+1} = -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t^2 - \left(\frac{\lambda}{\sigma}\right) x_t \varepsilon_{t+1}$$

Analagously, the short rate is still x_t

3.4 Strong Restrictions

Affine term structure models have strong predictions

- Since prices are linear in factors, a K-factor model generates a matrix of bond prices that has rank K. What this means is that given the vector of bond prices we can invert them to get a vector of states. Knowing the vector of states is everything that we need to predict the future values of states and, hence, future prices and returns Therefore, knowing bond yields already summarizes all information. This is counterfactual since Cochrane and Piazzesi (2005) show that including lagged forward rates improves predictive power.
- A way to deal with this is to introduce a hidden factor, factor that doesn't affect the term structure but it may be relevant to predict the future dynamics of other state variables. If we have a hidden factor, we can't longer invert the linear pricing equation and, therefore, the state variables provide additional information about future bond prices beyond information contained in bond prices.

4 Bond Pricing and Dynamics of Consumption Growth

Consider the covariation of future SDF and expectation of its future values

$$cov_t(m_{t+1}, E_{t+1} m_{t+2})$$

the expectation of a future SDF is positively related to bond prices and, hence, is negatively related to bon yields. This means that

$$cov_t(m_{t+1}, E_{t+1} m_{t+2}) \text{ and } cov_t(m_{t+1}, y_{1,t+1})$$

have opposite signs. Now we can think about how dynamics of SDF affects risk premium

- Suppose that shock to SDF covary positively with shock to expectations about future SDF meanins that SDF is *mean-reverting*. Then

$$cov_t(m_{t+1}, E_{t+1} m_{t+2}) < 0 \implies cov_t(m_{t+1}, y_{1,t+1}) > 0$$

  When there is a negative shock to SDF, the future SDF is expected to be larger. Hence, the bond prices rise and yields fall. In this case, bonds act as hedge and have negative risk premium

- Suppose now that shocks to SDF are persistent meaning that shocks to SDF covary positively with innovations to expectations of future SDF. Then

$$cov_t(m_{t+1}, E_{t+1}m_{t+2}) > 0 \implies cov_t(m_{t+1}, y_{1,t+1}) < 0$$

When there is a negative shock to SDF, the expectation of future SDF also fall $\implies$ bond prices fall and yields rise. Therefore, bonds prices fall in bad times and they command a positive risk premium.

We now relate this to models covered in Consumption CAPM part

Power Utility. In this model the short term rate is linear in the expected consumption growth

$$r_{f,t+1} = -\log(\delta) + \gamma E_t \Delta_{c,t+1} - \frac{\gamma^2}{2}\sigma_c^2$$

and innovations to the SDF is γ times the realized consumption growth. Hence, this model is equivalent to homoskedastic completely affine model where consumption growth follows an AR(1) process. Hence, we can think about a model where

$$E_t \Delta c_{t+1} = \mu + \phi E_{t-1}\Delta c_t + \sigma \varepsilon_t$$

$$\Delta c_{t+1} = E_t \Delta c_{t+1} + \frac{\lambda}{\sigma}\varepsilon_{t+1}$$

so that expected consumption growth follows an AR(1) and

- if $\lambda > 0$, then innovations to expected consumption growth are perfectly positively correlated with realizations of current consumption growth $\implies$ consumption growth is persistent $\implies$ bonds do well in bad times and do good in bad times as discussed earlier $\implies$ negative risk premium (hedges)
- if $\lambda < 0$, then innovations to expected consumption growth are perfectly negatively correlated with realizations of current consumption growth $\implies$ consumption growth is mean-reverting $\implies$ bonds perform good in good times and perform badly in bad times as discussed earlier $\implies$ positive risk premium

Epstein-Zin Preferences In Epstein-Zin preferences both current consumption and expectations about future consumption growth directly enter the SDF. As we know when $\gamma > \frac{1}{\psi}$ the SDF decreases on positive news about future consumption growth (see equation (**??**)). News about future consumption growth raises interest rates: we can see this for the 1-period rate in equation (**??**). This means that there is a positive covariance between bond returns even when the correlation between current consumption and expectations of future consumption is zero $\implies$ bonds have negative risk premium and they are hedges

Stochastic Volatility Increase in volatility stimulates increase in precautionary savings which drive interest rates down and bond prices up. Increase in volatility is bad news for a conservative investors with $\gamma > 1$, this also generates increase in SDF. Therefore, bond prices are positively correlated with SDF. Therefore, time-varying volatility amplifies the hedging property of bonds in Long-Run risks models.

Habit Model Campbell-Cochrane model has two offsetting effects

1. When consumption falls close to habit this raises the interest rate, since people want to borrow from the future where they will adjust to bad times. This drives bond prices down. *Another ways to see it is to note that in this model surplus ratio that drive marginal utility is mean reverting. Hence, bad shocks that move conusmption close to habit and decrease surplus also mean future growth of surplus and, hence, expected decline in marginal utility. Since* $r_{f,t+1} = -E_t m_{t+1} - \frac{1}{2}var_t(m_{t+1})$ *expected decrease in marginal mean increase in* $-E_t m_{t+1}$ *and, therefore, increase in rates and fall in bond prices*
2. On the other hand, when consumption falls close to habit the volatility of SDF goes up which generates precautionary savings that drives interest rates down (this is the $var_t(m_{t+1})$ element from above).

In the baseline calibration they cancel these two effects, but we can leave them

5 Permanent and Transitory Component of SDF

The definitions in the paper of Alvarez and Jermann (2005) are a bit misleading. The permanent component is a random walk component and transitory component is just a residual.

Consider a cumulative SDF (also called pricing kernel)

$$Q_t = M_1 \cdot \dots \cdot M_t$$

One period SDF that we are used to work with can be obtained as

$$M_{t+1} = \frac{Q_{t+1}}{Q_t}$$

With this notation the usual pricing equation for bonds is

$$P_{nt} = E_t[M_{t+1}M_{t+2}\dots M_{t+n}] = E_t\left[\frac{Q_{t+1}}{Q_t}\frac{Q_{t+2}}{Q_{t+2}}\dots\frac{Q_{t+n}}{Q_{t+n-1}}\right] = \frac{1}{Q_t}E_t[Q_{t+n}] \implies Q_t P_{nt} = E_t[Q_{t+n}]$$

Alvarez and Jermann show that under some conditions there exists β such that $\frac{E_t[Q_{t+n}]}{\beta^n}$ approaches a non-zero finite limit. Dividing the previous equation by such β we get

$$Q_t\frac{P_{nt}}{\beta^{t+n}} = E_t\left[\frac{Q_{t+n}}{\beta^{t+n}}\right]$$

They define a permanent component (rather a martingale component) of the pricing kernel Q_t as the limit

$$Q_t^P = \lim_{n\to\infty} E_t\left[\frac{Q_{t+n}}{\beta^{t+n}}\right]$$

and the transitory component is defined such that $Q_t = Q_t^T Q_t^P$.

- Permanent component has a property that $E_t Q_{t+1}^P = Q_t^P$. Correspondingly $E_t M_{t+1}^P = M_t^P$
- The price of inifinite-maturity bonds is related only to the transitory-component

$$\frac{Q_t^T}{Q_{t+1}^T} = \lim_{n\to\infty}\frac{\beta^{t+n}}{P_{nt}}\frac{P_{n-1,t+1}}{\beta^{t+n+1}} = \lim_{n\to\infty}\frac{\beta^{t+n}}{\beta^{t+n+1}}\lim_{n\to\infty}\frac{P_{nt}}{P_{n-1,t+1}} = 1 + R_{\infty,t+1}$$

$$\implies 1 + R_{\infty,t+1} = \frac{1}{M_{t+1}^T}$$

- Alvarez and Jermann (2005) also derive a relationship between the conditional entropy of the the SDF and its permanent component.

$$\begin{aligned}
L_t(M_{t+1}) &= \log E_t M_{t+1} - E_t \log(M_{t+1}) \\
&= \log E_t M_{t+1} - E_t \log(M_{t+1}^T M_{t+1}^P) \\
&= -r_{f,t+1} - E_t \log(M_{t+1}^P) - E_t \log(M_{t+1}^T) \\
&= -r_{f,t+1} - E_t \log(M_{t+1}^P) + \log E_t(M_{t+1}) + E_t r_{\infty,t+1} \\
&= -r_{f,t+1} + L_t(M_{t+1}^P) + E_t r_{\infty,t+1}
\end{aligned}$$

where I used that $E_t M_{t+1}^P = 1 \implies \log E_t(M_{t+1}^P = 0$ and $1 + R_{\infty,t+1} = \frac{1}{M_{t+1}^T}$

$$L_t(M_{t+1}^P) = L_t(M_{t+1}) - [E_t r_{\infty,t+1} - r_{f,t+1}]$$

This means that the conditional entropy of the permanent component is less than conditional entropy of the SDF itself when risk premium on infinite maturity bond is positive, i.e. infinite maturity bond has risk premia. It is lower when risk premia of an infinite maturity bond is negative. Recall that the bond has a negative risk premia when SDF has a predominantly positive autocorrelations so that innovations in current SDF are positively correlated with innovations in exectations of future SDF.

- We can combine this bound on the permanent component with entropy bound from chapter 4 that works

$$L_t(M_{t+1}) \geq E_t r_{j,t+1} - r_{f,t+1}$$

for any asset j. Then the relationship becomes

$$L_t(M^P_{t+1}) = L_t(M_{t+1}) - [E_t r_{\infty,t+1} - r_{f,t+1}] \geq E_t r_{j,t+1} - r_{\infty,t+1}$$

Even though we can't directly observe infinite maturity bonds, very long term bonds have much lower average return than equities an other risky assets. This suggests that the permanent component is quite important and significantly contributes to the volatility of the SDF.