Asset Pricing Notes. Chapter 6: Consumption Based Asset Pricing

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1 Lognormal Consumption with Power Utility

Consider the familiar CRRA (power, isoelastic) utility

$$u(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma}$$

$$u'(C_t) = C_t^{-\gamma}$$

SDF with such utility is

$$M_{t+1} = \delta \frac{u'(C_{t+1})}{u'(C_t)} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

and log SDF is

$$m_{t+1} = \log(\delta) - \gamma(c_{t+1} - c_t) = \log(\delta) - \gamma \Delta c_{t+1} \tag{1}$$

If consumption growth is distributed lognormally (Δc_{t+1} is lognormal) then SDF is also distributed lognormally. If we assume joint lognormality of asset returns and SDF we can use the expression for log risk premium from equation (**??**) to get the expression for risk premium

$$\begin{aligned}\mathbb{E}_t[r_{i,t+1}] - r_{f,t+1} + \frac{1}{2}\sigma_{it}^2 &= -\sigma_{imt} \\ &= -cov_t(r_{i,t+1}, m_{t+1}) \\ &= -cov_t(r_{i,t+1}, \log(\delta) - \gamma\Delta c_{t+1}) \\ &= \gamma cov_t(r_{i,t+1}, \Delta c_{t+1}) \equiv -\gamma\sigma_{ict}\end{aligned}$$

This implies that assets the covary strongly with consumption growth command a larger risk-premium. Using the log form of the pricing equation we can also determine the risk-free rate in this model

$$\begin{aligned}0 &= E_t r_{i,t+1} + E_t m_{t+1} + \frac{1}{2}\sigma_{it}^2 + \frac{1}{2}\sigma_{mt}^2 + \sigma_{imt} \\ 0 &= r_{f,t+1} + E_t m_{t+1} + \frac{1}{2}\sigma_{mt}^2 \\ r_{f,t+1} &= -E_t m_{t+1} - \frac{1}{2}\sigma_{mt}^2 \\ &= -\log(\delta) + \gamma E_t \Delta c_{t+1} - \frac{\gamma^2}{2}\sigma_{ct}^2\end{aligned}$$

Three Puzzles The solution of this model is very simple, however, it has many problems

1. **Equity Premium Puzzle** High average return on equities combined with low σ_{ict} implies very large γ – coefficient of relative risk aversion

2. **Risk Free Rate Puzzle** Even if we allow γ to be large we are going to have problems with the risk free rate. Consider the determinants of the risk free rate

$$r_{f,t+1} = \underbrace{-\log(\delta)}_{\text{time preference}} + \underbrace{\gamma E_t \Delta c_{t+1}}_{\text{intertemporal substitution}} - \underbrace{\frac{\gamma^2}{2}\sigma_{ct}^2}_{\text{precautionary savings}}$$

   Since the volatility of consumption growth σ_{ct} is small for small values of γ the intertemporal substitution term dominates. However, when γ is large the precautionary savings term dominates: investors react very strongly to increases in uncertainty about future consumption growth by increasing savings and driving down risk-free interest rate.

3. **Equity Volatility Puzzle** Stock returns are much more volatile than consumption growth. This is puzzling if we view the market portfolio as the claim to the economy that gives consumption as dividends. Essentially it is the same argument that Shiller (1988) used to claim that volatility of prices is not justified by the volatility of dividends.

2 Beyond Lognormality

The first approach that we use to deal with the puzzles is changing the process for consumption growth by introducing a left tail – disasters. The consumption process still tays iid. We are going to use cumulant generating function as a tractable way to calculate all the variables that we want in closed form

2.1 General Case of Cumulant Pricing

Consider an asset that pays dividend $D_t = C_t^\lambda$. Values of $\lambda > 1$ might be thought as leverage, i.e. the stock market portfolio pays a levered consumption as its dividends. Price of an asset is given by

$$\begin{aligned}
P_t &= E_t \sum_{j=1}^{\infty} \delta^j \left(\frac{C_{t+j}}{C_t}\right)^{-\gamma} D_{t+j} \\
&= E_t \sum_{j=1}^{\infty} \delta^j \left(\frac{C_{t+j}}{C_t}\right)^{-\gamma} C_{t+j}^\lambda \\
&= C_t^\lambda E_t \sum_{j=1}^{\infty} \delta^j \left(\frac{C_{t+j}}{C_t}\right)^{-\gamma} \frac{C_{t+j}^\lambda}{C_t^\lambda} \\
&= D_t E_t \sum_{j=1}^{\infty} \delta^j \left(\frac{C_{t+j}}{C_t}\right)^{\lambda-\gamma}
\end{aligned}$$

Define $\delta = \exp(-r^*)$ and $g_{t+1} = \Delta c_{t+1}$ – log consumption change. Then

$$\begin{aligned}
P_t &= D_t E_t \sum_{j=1}^{\infty} \delta^j \left(\exp(\sum_{k=1}^{j} \Delta c_{t+k})\right)^{\lambda-\gamma} \\
&= D_t E_t \sum_{j=1}^{\infty} \exp(-r^* j) \prod_{k=1}^{j} \exp((\lambda-\gamma) g_{t+j}) \\
&= D_t \sum_{j=1}^{\infty} \exp(-r^* j) E_t \prod_{k=1}^{j} \exp((\lambda-\gamma) g_{t+j})
\end{aligned}$$

using the assumption that consumption growth g_t is iid we can write

$$\begin{aligned}
P_t &= D_t \sum_{j=1}^{\infty} \exp(-r^* j) \prod_{k=1}^{j} E_t \exp((\lambda-\gamma) g_{t+k}) \\
&= D_t \sum_{j=1}^{\infty} \exp(-r^* j) \left[E_t \exp((\lambda-\gamma) g)\right]^j
\end{aligned}$$

Recall that cumulant generating function is

$$\mathbf{c}(\theta, x) = \log E \exp(\theta x) \implies E \exp(\theta x) = \exp(\mathbf{c}(\theta, x))$$

write price of the asset in terms of the cumulant generating function with $\theta = \lambda - \gamma$

$$\begin{aligned}
P_t &= D_t \sum_{j=1}^{\infty} \exp(-r^* j) \exp(\mathbf{c}(\lambda-\gamma, g))^j \\
&= D_t \sum_{j=1}^{\infty} \exp(-(r^* - \mathbf{c}(\lambda-\gamma, g)) j) \\
&= D_t \frac{\exp(-(r^* - \mathbf{c}(\lambda-\gamma, g)))}{1 - \exp(-(r^* - \mathbf{c}(\lambda-\gamma, g)))} \qquad \implies \frac{D_t}{P_t} = \exp(r^* - \mathbf{c}(\lambda-\gamma, g))) - 1 \implies \frac{D_t}{P_t} = \frac{D_{t+1}}{P_{t+1}}
\end{aligned}$$

Gross return on the asset is equal to

$$\begin{aligned}
1+R_{t+1} &= \frac{D_{t+1}+P_{t+1}}{P_t} = \frac{D_{t+1}}{P_{t+1}}\frac{P_{t+1}}{P_t} + \frac{P_{t+1}}{P_t} = \frac{P_{t+1}}{P_t}\left(1+\frac{D_{t+1}}{P_{t+1}}\right) \\
&= \frac{P_{t+1}}{P_t}\left(1+\frac{1-\exp(-(r^*-\mathbf{c}(\lambda-\gamma,g))j)}{\exp(-(r^*-\mathbf{c}(\lambda-\gamma,g))j)}\right) \\
&= \frac{D_{t+1}}{D_t}\exp(r^*-\mathbf{c}(\lambda-\gamma,g))) \\
&= \frac{C^\lambda_{t+1}}{C^\lambda_t}\exp(r^*-\mathbf{c}(\lambda-\gamma,g))) \\
&= \underbrace{\exp(\lambda g_{t+1})}_{\text{random}}\underbrace{\exp(r^*-\mathbf{c}(\lambda-\gamma,g)))}_{\text{non-random}}
\end{aligned}$$

Expected return is, therefore

$$\begin{aligned}
1+E_tR_{t+1} &= E_t\exp(\lambda g_{t+1})\exp(r^*-\mathbf{c}(\lambda-\gamma,g))) \\
&= \exp(\mathbf{c}(\lambda,g))\exp(r^*-\mathbf{c}(\lambda-\gamma,g))) \\
&= \exp(r^*-\mathbf{c}(\lambda-\gamma,g))+\mathbf{c}(\lambda,g)) \\
\implies er(\lambda)\equiv\log(1+E_tR_{t+1}) &= \underbrace{r^*-\mathbf{c}(\lambda-\gamma,g))}_{=\log(1+D/P)}+\mathbf{c}(\lambda,g)
\end{aligned}$$

Risk Free Rate Risk free asset always pays $D_t = 1 \implies \lambda = 0$. Then

$$r_f = r^*-\mathbf{c}(-\gamma,g))+\mathbf{c}(0,g)) = r^*-\mathbf{c}(-\gamma,g)) = r^*-\sum_{n=1}^{\infty}\frac{\kappa_n(-\gamma)^n}{n!}$$

Claim on Overall Consumption The claim on overall consumption pays $D_t = C_t \implies \lambda = 1$. First note that as usual $\log(1+E_tR_{t+1}) = r_{t+1}+\frac{1}{2}\sigma^2_{it}$. Then the risk premium

$$\begin{aligned}
r_{t+1}-r_{f,t+1}+\frac{1}{2}\sigma^2_{it} &= \log(1+E_tR_{t+1})-r^*+\mathbf{c}(-\gamma,g)) \\
&= r^*-\mathbf{c}(1-\gamma,g))+\mathbf{c}(1,g)-r^*+\mathbf{c}(-\gamma,g)) \\
&= -\mathbf{c}(1-\gamma,g))+\mathbf{c}(1,g)+\mathbf{c}(-\gamma,g)) \\
&= \sum_{n=1}^{\infty}\frac{\kappa_n}{n!}\left[1+(-\gamma)^n-(1-\gamma)^n\right]
\end{aligned} \tag{2}$$

Notice that the risk premium includes an infinite number of cumulants, i.e. it depends on all higher moments of the distribution of the growth rate of consumption. One can naively assume that the contibution of higher moments is negligible and decide to truncate the expression say at $n = 4$. However, Martin (2013) shows that for realistic examples (inluding the rare disaster application discussed below) the convergence in the cumulant generating function is quite slow. Therefore, truncation will largely underestimate the equity premium.

Case of Lognormal Consumption In the case of lognormal consumption growth $\kappa_1 = \mu, \kappa_2 = \sigma^2_c, \kappa_3 = \kappa_4 = \cdots = 0$ so that we have

$$\begin{aligned}
r_{t+1}-r_{f,t+1}+\frac{1}{2}\sigma^2_{it} &= \mu\left[1+(-\gamma)-(1-\gamma)\right]+\frac{\sigma^2_c}{2}\left[1+(-\gamma)^2-(1-\gamma)^2\right] \\
&= \frac{\sigma^2_c}{2}\left[1+\gamma^2-1+2\gamma-\gamma^2\right] \\
&= \gamma\sigma^2_c
\end{aligned}$$

which is exactly the same expression that we had before.

2.2 Application to Rare Disasters

In the case analyzed in Barro (2006) with continuous time jump diffusion for log consumption the cumulant generating function takes the following form

$$\mathbf{c}(\theta, g) = \mu\theta + \frac{1}{2}\sigma^2\theta^2 + \omega\left(E[\exp(-\theta x)] - 1\right)$$

where (μ, σ) are diffusion parameters, ω is the intensity of a rare disaster and x – random size of a rare disaster. Under $x \sim \mathcal{N}(m, s^2)$ we have

$$\mathbf{c}(\theta, g) = \mu\theta + \frac{1}{2}\sigma^2\theta^2 + \omega\left(\exp\left(-\theta\mu + \frac{1}{2}\theta^2 s^2\right) - 1\right)$$

Now we can apply the expression for risk premium in equation (2). First note that we can split the risk premium into two parts: one that comes from the diffusion and one that comes from disasters. Let's deal with the diffusion part

$$\begin{aligned} eqp^{diffusion} &= -\mathbf{c}(1-\gamma, g)) + \mathbf{c}(1, g) + \mathbf{c}(-\gamma, g)) \\ &= -\mu(1-\gamma) - \frac{1}{2}(1-\gamma)^2\sigma^2 + \mu + \frac{1}{2}\sigma^2 - \mu\gamma + \frac{1}{2}\gamma^2\sigma^2 \\ &= -\frac{1}{2}(1-\gamma)^2\sigma^2 + \frac{1}{2}\sigma^2 + \frac{1}{2}\gamma^2\sigma^2 \\ &= \gamma\sigma^2 \end{aligned}$$

which not surprisingly is exactly the same expression as before. The part of equity premium that comes from disasters is

$$\begin{aligned} eqp^{disaster} &= -\mathbf{c}(1-\gamma, g)) + \mathbf{c}(1, g) + \mathbf{c}(-\gamma, g)) \\ &= -\omega\left(E[\exp(-(1-\gamma)x)] - 1\right) + \omega\left(E[\exp(-x)] - 1\right) + \omega\left(E[\exp(\gamma x)] - 1\right) \\ &= \omega E\left[\exp(-x)\left(1 - \exp(\gamma x)\right) - \left(1 - \exp(\gamma x)\right)\right] \\ &= \omega E\left[\exp(-x)\left(1 - \exp(\gamma x)\right) - \left(1 - \exp(\gamma x)\right)\right] \\ &= \omega E\left[(1 - \exp(\gamma x)])\left(\exp(-x) - 1\right)\right] \\ &= \omega E\left[\left(B^{-\gamma} - 1\right)(1 - B)\right] \end{aligned}$$

Where $(1 - B)$ is the loss of consumption during a disaster and $(B^{-\gamma} - 1)$ is the increase in utility during a disaster. We can combine both components to get the final expression for the risk premium in a baseline disaster model

$$eqp = \gamma\sigma^2 + \omega E\left[\left(B^{-\gamma} - 1\right)(1 - B)\right] \tag{3}$$

3 Epstein-Zin Preferences

CRRA utility imposes a very tight link between the relative risk aversion and the elasticity of intertemporal substitution: they are reciprocal of each other. Because of this we can't increase risk aversion to solve the equity premium, since it will give a very low value of EIS $\implies$ investors are not willing to allow consumption vary across time periods $\implies$ implausible behavior of the risk free rate. Epstein and Zin (1989, 1991) recursive utility allows to separate the two and solve risk premium puzzle by ramping up risk aversion without changing EIS.

3.1 Simple example

In order to understand how EZ utility separates RA and EIS consider a simple two period example where agent consumes C_0 at $t = 0$ and stochastic C_1 at $t = 1$. Given stochastic C_1 you can think about what level of **deterministic** consumption $\overline{C}_1$ will make you indifferent between receiving $\overline{C}_1$ and C_1. Denote this mapping $\overline{C}_1 = m(C_1)$ where $m(\cdot)$ is the certainty equivalent function. This function encodes agent's attitudes toward risk.

Now we can aggregate agents lifetime consumption with some function $W(C_0, \overline{C}_1)$. This function measures agent's attitude toward intertemporal substitution and not the risk aversion component. Hence, we arrive at the following utility function over the consumption in two periods

$$V = W(C_0, m(C_1))$$

that has a clear separation between risk aversion and intertemporal substitution forces.

3.2 General Case

Epstein Zin preferences generalize the above logic to many periods. They have the following form

$$U_t = f(C_t, \mu(U_{t+1}))$$

where $\mu(\cdot)$ is a certainty equivalent function and $f(\cdot,\cdot)$ is a time aggregator. They a assume a particular form for both

$$CRRA : \mu(U_{t+1}) = (E_t U_{t+1}^{1-\gamma})^{\frac{1}{1-\gamma}}$$
$$CES : f(x, y) = ((1-\delta)x^{1-\frac{1}{\psi}} + \delta y^{1-\frac{1}{\psi}})^{\frac{1}{1-1/\psi}}$$

to get

$$\begin{aligned} U_t &= \left[(1-\delta)C_t^{1-1/\psi} + \delta\left(E_t U_{t+1}^{1-\gamma}\right)^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}} \\ &= \left[(1-\delta)C_t^{\theta(1-\gamma)} + \delta\left(E_t U_{t+1}^{1-\gamma}\right)^{\theta}\right]^{\frac{\theta}{1-\gamma}}, \text{ where } \theta = \frac{1-1/\psi}{1-\gamma} \end{aligned} \tag{4}$$

Some Properties of Epstein Zin Utility

1. As discussed for power utility we have $\gamma = \frac{1}{\psi} \implies \theta = 1$. Plug this into equation (4) and iterate forward

$$\begin{aligned} U_t &= \left[(1-\delta)C_t^{(1-\gamma)} + \delta\left(E_t U_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{1-\gamma}} \\ &= \left[(1-\delta)C_t^{(1-\gamma)} + \delta(1-\delta)E_t C_{t+1} + \delta^2 E_t U_{t+2}^{1-\gamma}\right]^{\frac{1}{1-\gamma}} \\ &= \left[(1-\delta)\left(C_t^{(1-\gamma)} + \delta E_t C_{t+1} + \delta^2 E_t C_{t+2}^{1-\gamma} + \dots\right)\right]^{\frac{1}{1-\gamma}} \end{aligned}$$

   Take a monotonic transformation of U_t to arrive at

$$V_t = \frac{1-\delta}{1-\gamma}U_t^{1-\gamma} = E_t \sum_{j=1}^{\infty} \delta^j \frac{C_{t+j}^{1-\gamma}}{1-\gamma}$$

   which is a standard CRRA lifetime utility.

2. In case of $\psi \to 1$ the CES aggregator converges to Cobb-Douglas aggregator

$$U_t = C_t^{1-\delta}\left(E_t[U_{t+1}^{1-\gamma}]\right)^{\frac{\delta}{1-\gamma}}$$

   Take logs

$$\log(U_t) = (1-\delta)\log(C_t) + \frac{\delta}{1-\gamma}\log\left(E_t[U_{t+1}^{1-\gamma}]\right)$$

   and define $V = \frac{1}{1-\delta}\log(U) \implies U = \exp((1-\delta)V)$ to get

$$\begin{aligned} V_t &= \log(C_t) + \frac{\delta}{(1-\gamma)(1-\delta)}\log E_t \exp((1-\gamma)(1-\delta)V_{t+1}) \\ &= \log(C_t) - \delta\lambda \log E_t \exp\left(-\frac{V_{t+1}}{\lambda}\right) \end{aligned}$$

   This is called **risk-sensitive recursion**.

3. For $\psi = 1$ consumption-wealth ratio is constant. This was the case with CRRA utility when $\gamma = 1/\psi = 1$.
4. For $\gamma = 1$, investment horizon doesn't affect portfolio choice. *More on this in chapter 9 on intertemporal risk*
5. Epstein-Zin consumers are not indifferent to the timing of uncertainty resolution. When $\gamma > \frac{1}{\psi}$ agents prefer earlier resolution of uncertainty. Epstein, Farhi and Strzalecki argue that long-run risk calibration of Epstein-Zin implies implausibly high willingness to pay for uncetainty resolution.

3.3 SDF for Epstein-Zin

Here we are going to derive the SDF for Epstein-Zin utility[1]. This is going to be a tricky endeavor since change in future consumption affect current utility as well because per-period utility functions are not time separable. However, the argument remain similar just with more derivations. Suppose that the history of states is $s^t = (\dots, s_{t-2}, s_{t-1}, s_t)$ and we consider the effect of reducing current consumption $c_t(s^t)$ by Δ and buying A-D security with payoff in state s_{t+1} and its effect on out utility. Reducing consumption reduces utility by

$$\frac{\partial V_t}{\partial c_t(s^t)}\Delta$$

Additional consumption from A-D security gives

$$\frac{\Delta}{q(s^t \to s_{t+1})}\frac{\partial V_t}{\partial c_{t+1}(s^t, s_{t+1})}$$

Notice, that there is no probability in this expression since the expectation over future outcomes is already takes care of in V_t. In equilibrium the agent should be indifferent between the two:

$$\frac{\partial V_t}{\partial c_t(s^t)}\Delta = \frac{\Delta}{q(s^t \to s_{t+1})}\frac{\partial V_t}{\partial c_{t+1}(s^t, s_{t+1})}$$

$$q(s^t \to s_{t+1}) = \frac{\partial V_t/\partial c_{t+1}(s^t, s_{t+1})}{\partial V_t/\partial c_t(s^t)}$$

Now we just need to calculate each of the partial derivatives. To save space I will use $\mu(\cdot)$ to denote the certainty equivalent function and $f(x_1, x_2)$ to denote the time aggregator. Start with numerator

$$\begin{aligned}
\frac{\partial V_t}{\partial c_{t+1}(s^t, s_{t+1})} &= \frac{\partial f(C_t, \mu(V_{t+1}))}{\partial c_{t+1}(s^t, s_{t+1})} \\
&= f_2(C_t, \mu(V_{t+1}))\frac{\partial \mu(V_{t+1})}{\partial V_{t+1}(s^t, s_{t+1})}\frac{\partial V_{t+1}(s^t, s_{t+1})}{\partial c_{t+1}(s^t, s_{t+1})} \\
&= f_2(C_t, \mu(V_{t+1}))\frac{\partial \mu(V_{t+1})}{\partial V_{t+1}(s^t, s_{t+1})} f_1(c_{t+1}(s^t, s_{t+1}), \mu_t(V_{t+2}))
\end{aligned}$$

Consider each of the terms:

$$\begin{aligned}
f_2(C_t, \mu(V_{t+1})) &= \frac{1}{1-1/\psi}\left((1-\delta)C_t^{1-1/\psi} + \delta\mu(V_{t+1})^{1-1/\psi}\right)^{\frac{1}{1-1/\psi}-1}\delta\mu(V_{t+1})^{-1/\psi} \\
&= \left((1-\delta)C_t^{1-1/\psi} + \delta\mu(V_{t+1})^{1-1/\psi}\right)^{\frac{1/\psi}{1-1/\psi}}\delta\mu(V_{t+1})^{-1/\psi} \\
&= f(c_t, \mu(V_{t+1}))^{1/\psi}\delta\mu(V_{t+1})^{-1/\psi} = \delta\mu(V_{t+1})^{-1/\psi}V_t^{1/\psi}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mu(V_{t+1})}{\partial V_{t+1}(s^t, s_{t+1})} &= \frac{\partial}{\partial V_{t+1}(s^t, s_{t+1})}\left(\sum_{s_{t+1}|s^t}\pi(s^t \to s_{t+1})V_{t+1}(s^t, s_{t+1})\right)^{\frac{1}{1-\gamma}} \\
&= \frac{1}{1-\gamma}\left(\sum_{s_{t+1}|s^t}\pi(s^t \to s_{t+1})V_{t+1}(s^t, s_{t+1})^{1-\gamma}\right)^{\frac{1}{1-\gamma}-1}\pi(s^t \to s_{t+1})(1-\gamma)V_{t+1}(s^t, s_{t+1})^{-\gamma} \\
&= \mu(V_{t+1})^{\gamma}\pi(s^t \to s_{t+1})V_{t+1}(s^t, s_{t+1})^{-\gamma}
\end{aligned}$$

[1] This derivation is based on Francois Gourio notes that can be found on his personal page

$$f_1(c_{t+1}(s^t,s_{t+1}),\mu_t(V_{t+2})) = \frac{1}{1-1/\psi}\left((1-\delta)c_{t+1}^{1-1/\psi} + \delta\mu_{t+1}(V_{t+2})^{1-1/\psi}\right)^{\frac{1}{1-1/\psi}-1}(1-\delta)\left(1-\frac{1}{\psi}\right)c_{t+1}(s^t,s_{t+1})^{-1/\psi}$$
$$= V_{t+1}^{1/\psi}(1-\delta)c_{t+1}(s^t,s_{t+1})^{-1/\psi}$$

Denominator is now straightforward

$$\frac{\partial V_t}{\partial C_t(s^t)} = V_t^{1/\psi}C_t^{-1/\psi}(1-\delta)$$

Combine all expressions to get

$$\frac{\partial V_t/\partial c_{t+1}(s^t,s_{t+1})}{\partial V_t/\partial c_t(s^t)} = \frac{\delta\mu(V_{t+1})^{-1/\psi}V_t^{1/\psi}\times\mu(V_{t+1})^{\gamma}\pi(s^t\to s_{t+1})V_{t+1}(s^t,s_{t+1})^{-\gamma}\times V_{t+1}^{1/\psi}(1-\delta)c_{t+1}(s^t,s_{t+1})^{-1/\psi}}{V_t^{1/\psi}C_t^{-1/\psi}(1-\delta)}$$
$$= \pi(s^t\to s_{t+1})\left[\mu(V_{t+1})^{-1/\psi}\cdot\mu(V_{t+1})^{\gamma}V_{t+1}(s^t,s_{t+1})^{-\gamma}V_{t+1}^{1/\psi}\right]\cdot\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}$$
$$= \pi(s^t\to s_{t+1})\left[\mu(V_{t+1})^{\gamma-1/\psi}\cdot V_{t+1}^{-(\gamma-1/\psi)}\right]\cdot\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}$$
$$= \pi(s^t\to s_{t+1})\left(\frac{V_{t+1}}{\mu_t(V_{t+1})}\right)^{-\left(\gamma-\frac{1}{\psi}\right)}\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}$$

Now recall that SDF is

$$M(s^t,s_{t+1}) \equiv \frac{q(s^t\to s_{t+1})}{\pi(s^t\to s_{t+1})} = \delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left(\frac{V_{t+1}}{\mu_t(V_{t+1})}\right)^{-\left(\gamma-\frac{1}{\psi}\right)}$$

3.4 Working with Epstein-Zin SDF

Rearrange the SDF to get

$$\begin{aligned} M_{t+1} &= \delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{\gamma-\frac{1}{\psi}}\left(\frac{V_{t+1}}{\mu_t(V_{t+1})}\right)^{-\left(\gamma-\frac{1}{\psi}\right)} \\ &= \delta\left(\frac{\mu_t(V_{t+1})}{C_t}\right)^{\gamma-\frac{1}{\psi}}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{V_{t+1}}{C_{t+1}}\right)^{-\left(\gamma-\frac{1}{\psi}\right)} \\ &= \delta\Gamma_t\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{V_{t+1}}{C_{t+1}}\right)^{-\left(\gamma-\frac{1}{\psi}\right)} \end{aligned} \tag{5}$$

Notice that Γ_t is known at time t. Also when $\theta = \frac{1}{\psi}$ the term with continuation utility disappears and we are back to the CRRA SDF. In what follows it will be very useful to work with innovations in log variables. Denote innovation as $\tilde{x}_{t+1} = x_{t+1} - E_t x_{t+1}$. Now consider innovations to the log SDF from equation (5).

$$\tilde{m}_{t+1} = -\gamma\tilde{c}_{t+1} - \left(\gamma - \frac{1}{\psi}\right)(\tilde{u}_{t+1} - \tilde{c}_{t+1}) \tag{6}$$

Both shocks to consumption and to the continuation utility are priced. This will help us to generate risk premium in the long run risk model. However, equation (6) is not particularly useful when we take it to the data since it involves continuation utility that is not observed. Therefore, we need to express this in a different way

3.5 Substituting out Continuation Utility

We can "substitute" out continuation utility in the SDF using intertemporal budget constraint

$$W_{t+1} = (W_t - C_t)(1 + R_{w,t+1})$$

where $1+R_{w,t+1}$ is the return on the wealth portfolio. The main step in this derivation is to conjecture the following form for the utility function $U_t(W_t,X_t)=\phi(X_t)W_t=\phi_t W_y$ where X_t are some state variables. This conjecture is natural since the time aggregator and certainty equivalent functions are homogenous of degree 1. Use this conjecture to rewrite the utility function as

$$U(W_t,X_t)=\max_{C_t,v_t}\left\{(1-\delta)C_t^\rho+\delta\left[E_t[\phi_{t+1}^\alpha(1+R_{w,t+1})^\alpha]\right]^{\rho/\alpha}(W_t-C_t)^\rho\right\}^{1/\rho}$$

where I define $\rho=1-1/\psi$ and $\alpha=1-\gamma$ and v_t is the vector of portfolio weights so that $R_{w,t+1}=v_t'R_{t+1}$ (we will get back to it). Define $\mu_t^*\equiv\left[E_t[\phi_{t+1}^\alpha(1+R_{w,t+1})^\alpha]\right]^{1/\alpha}$ The FOC w.r.t consumption is

$$(1-\delta)\rho C_t^{\rho-1}-\delta(\mu_t^*)^\rho\rho(W_t-C_t)^{\rho-1}=0$$

This expression implies a consumption rule of the form $C_t=\psi W_t$. Use this to derive the expression for μ_t^*:

$$\begin{aligned}\frac{(1-\delta)}{\delta}\psi_t^{\rho-1}W_t^{\rho-1}&=(\mu_t^*)^\rho(W_t-\psi_tW_t)^{\rho-1}\\ \frac{(1-\delta)}{\delta}\psi_t^{\rho-1}&=(\mu_t^*)^\rho(1-\psi_t)^{\rho-1}\\ (\mu_t^*)^\rho&=\left(\frac{\psi_t}{1-\psi_t}\right)^{\rho-1}\frac{(1-\delta)}{\delta}\end{aligned}$$

Now plug this back into the value function U_t

$$\begin{aligned}U(W_t,X_t)&=[(1-\delta)C_t^\rho+\delta(\mu_t^*)^\rho(W_t-C_t)^\rho]^{1/\rho}\\ &=\left[(1-\delta)(\psi_tW_t)^\rho+\delta\left(\frac{\psi_t}{1-\psi_t}\right)^{\rho-1}\frac{(1-\delta)}{\delta}(1-\psi_t)^\rho W_t^\rho\right]^{1/\rho}\\ &=(1-\delta)^{1/\rho}\left[(\psi_t)^\rho+\psi_t^{\rho-1}(1-\psi_t)\right]^{1/\rho}W_t\\ &=\underbrace{(1-\delta)^{1/\rho}\psi_t^{\frac{\rho-1}{\rho}}}_{\phi_t}W_t\\ &=\phi_tW_t\end{aligned}$$

Hence, the conjecture that the value function is linear in wealth W_t is verified.

Now we're going to express the consumption FOC in terms of observable variables (and later it will help us to derive the SDF):

$$\frac{(1-\delta)}{\delta}\psi_t^{\rho-1}=(\mu_t^*)^\rho(1-\psi_t)^{\rho-1}$$

$$\psi_t^{\rho-1}=\frac{\delta}{(1-\delta)}E_t\left[\phi_{t+1}^\alpha(1+R_{W,t+1})^\alpha\right]^{\rho/\alpha}(1-\psi_t)^{\rho-1}\text{ (used definition of }\mu_t^*)$$

$$\psi_t^{\rho-1}=\frac{\delta}{(1-\delta)}E_t\left[(1-\delta)^{\alpha/\rho}\psi_{t+1}^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^\alpha\right]^{\rho/\alpha}(1-\psi_t)^{\rho-1}\text{ (used expression for }\phi\text{ in terms of }\psi)$$

$$\psi_t^{\rho-1}=\delta E_t\left[\left(\frac{C_{t+1}}{W_{t+1}}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^\alpha\right]^{\rho/\alpha}(1-\psi_t)^{\rho-1}\text{ (used definition of }\psi)$$

$$\psi_t^{\rho-1}=\delta E_t\left[\left(\frac{C_{t+1}}{(W_t-C_t)(1+R_{W,t+1})}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^\alpha\right]^{\rho/\alpha}(1-\psi_t)^{\rho-1}\text{ (used definition of }\psi\text{ and budget constraint)}$$

$$\psi_t^{\rho-1}=\delta E_t\left[\left(\frac{C_{t+1}}{W_t-C_t}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha/\rho}\right]^{\rho/\alpha}(1-\psi_t)^{\rho-1}$$

$$\left(\frac{C_t}{W_t}\right)^{\rho-1} = \delta E_t\left[\left(\frac{C_{t+1}}{W_t - C_t}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha/\rho}\right]^{\rho/\alpha}\left(\frac{W_t - C_t}{W_t}\right)^{\rho-1}$$

$$C_t^{\rho-1} = \delta E_t\left[(C_{t+1})^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha/\rho}\right]^{\rho/\alpha}$$

$$1 = \delta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha/\rho}\right]^{\rho/\alpha}$$

There are two ways to proceed from this point on. First, is to use the SDF that we found above in the case of complete markets. Second, is to proceed with dynamic programming approach and solve the portfolio choice problem. Below I will show both of these ways

Complete Markets Approach Use the SDF that we derive in previous section and substitute $U_{t+1} = \phi_{t+1}W_{t+1}$

$$\begin{aligned}
M_{t+1} &= \delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left(\frac{V_{t+1}}{\mu_t(V_{t+1})}\right)^{-(\gamma-\frac{1}{\psi})} \\
&= \delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left(\frac{\phi_{t+1}W_{t+1}}{E_t\left[(\phi_{t+1}W_{t+1})^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{-(\gamma-\frac{1}{\psi})} \\
&= \delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left(\frac{(1-\delta)^{\frac{1}{1-1/\psi}}\psi_{t+1}^{\frac{1/\psi}{1-1/\psi}}(W_t - C_t)(1+R_{W,t+1})}{E_t\left[\left((1-\delta)^{\frac{1}{1-1/\psi}}\psi_{t+1}^{\frac{1/\psi}{1-1/\psi}}(W_t - C_t)(1+R_{W,t+1})\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{-(\gamma-\frac{1}{\psi})} && \text{(Subst. } \phi_{t+1} \text{ in terms of } \psi_{t+1}) \\
&= \delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left(\frac{\left(\frac{C_{t+1}}{(W_t-C_t)(1+R_{W,t+1})}\right)^{\frac{1/\psi}{1-1/\psi}}(1+R_{W,t+1})}{E_t\left[\left(\left(\frac{C_{t+1}}{(W_t-C_t)(1+R_{W,t+1})}\right)^{\frac{1/\psi}{1-1/\psi}}(1+R_{W,t+1})\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{-(\gamma-\frac{1}{\psi})} && \text{(Plugged } \psi_{t+1} = \frac{C_{t+1}}{W_{t+1}}) \\
&= \delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left(\frac{\left(\frac{C_{t+1}}{C_t}\right)^{\frac{1/\psi}{1-1/\psi}}(1+R_{W,t+1})^{\frac{1}{1-1/\psi}}}{E_t\left[\left(\left(\frac{C_{t+1}}{C_t}\right)^{\frac{1/\psi}{1-1/\psi}}(1+R_{W,t+1})^{\frac{1}{1-1/\psi}}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{-(\gamma-\frac{1}{\psi})}
\end{aligned}$$

Now rearrange the FOC for consumption that we derived above to get

$$1 = \delta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha/\rho}\right]^{\rho/\alpha} \implies \delta^{-\frac{1}{1-1/\psi}} = E_t\left[\left(\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1/\psi}{1-1/\psi}}(1+R_{W,t+1})^{\frac{1}{1-1/\psi}}\right)^{1-\gamma}\right]$$

Plug this expression back to the SDF to obtain the final expression

$$
\begin{aligned}
M_{t+1} &= \delta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}} \left(\frac{\left(\frac{C_{t+1}}{C_t}\right)^{\frac{1/\psi}{1-1/\psi}} (1+R_{W,t+1})^{\frac{1}{1-1/\psi}}}{\delta^{-\frac{1}{1-1/\psi}}} \right)^{-\left(\gamma-\frac{1}{\psi}\right)} \\
&= \delta^{1-\frac{1}{1-1/\psi}\left(\gamma-\frac{1}{\psi}\right)} \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}} \left(\left(\frac{C_{t+1}}{C_t}\right)^{\frac{1/\psi}{1-1/\psi}} (1+R_{W,t+1})^{\frac{1}{1-1/\psi}} \right)^{-\left(\gamma-\frac{1}{\psi}\right)} \\
&= \delta^{\frac{1-\gamma}{1-1/\psi}} \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1-\gamma}{\psi-1}} \left(\frac{1}{1+R_{W,t+1}}\right)^{1-\frac{1-\gamma}{1-1/\psi}}
\end{aligned}
$$

Dynamic Programming Approach Now we consider portfolio choice problem and consider the allocation of residual wealth $W_t - C_t$ to assets. Asset doesn't affect current consumption in the optimum. Therfore, the problem of asset allocation is

$$\max_{v_t} E_t[U^\alpha_{t+1}]^{1/\alpha} \text{ s.to } v_t'\mathbf{1} = 1$$

$$\max_{v_t} E_t[\phi^\alpha_{t+1} W^\alpha_{t+1}]^{1/\alpha} \text{ s.to } v_t'\mathbf{1} = 1$$

$$\max_{v_t} E_t[\phi^\alpha_{t+1}(W_t - C_t)(1+R_{W,t+1})^\alpha]^{1/\alpha} \text{ s.to } v_t'\mathbf{1} = 1$$

$$\max_{v_t} E_t[\phi^\alpha_{t+1}(1+R_{W,t+1})^\alpha]^{1/\alpha} \text{ s.to } v_t'\mathbf{1} = 1$$

$$\max_{v_t} E_t[(\phi_{t+1} v_t'(1+R_{t+1}))^\alpha]^{1/\alpha} \text{ s.to } v_t'\mathbf{1} = 1$$

Lagrangian for this problem is

$$\mathcal{L} = E_t[(\phi_{t+1} v_t'(1+R_{t+1}))^\alpha]^{1/\alpha} - \lambda(v_t'\mathbf{1} - 1)$$

and the FOC w.r.t. $v_t^{(i)}$ (ith element of vector of portfolio weights v_t) is

$$\frac{1}{\alpha} E_t[(\phi_{t+1} v_t'(1+R_{t+1}))^\alpha]^{1/\alpha-1} E_t\left[\alpha(\phi_{t+1} v_t'(1+R_{t+1}))^{\alpha-1}\phi_{t+1}(1+R_{i,t+1})\right] = \lambda$$

$$E_t[(\phi_{t+1}(1+R_{W,t+1}))^\alpha]^{1/\alpha-1} E_t\left[\phi^\alpha_{t+1}(1+R_{W,t+1})^{\alpha-1}(1+R_{i,t+1})\right] = \lambda$$

Multiply on both sides by $v_t^{(i)}$

$$E_t[(\phi_{t+1}(1+R_{W,t+1}))^\alpha]^{1/\alpha-1} E_t\left[\phi^\alpha_{t+1}(1+R_{W,t+1})^{\alpha-1} v_t^{(i)}(1+R_{i,t+1})\right] = v_t^{(i)}\lambda$$

and sum across all assets

$$E_t[(\phi_{t+1}(1+R_{W,t+1}))^\alpha]^{1/\alpha-1} E_t\left[\phi^\alpha_{t+1}(1+R_{W,t+1})^{\alpha-1}(1+R_{W,t+1})\right] = \lambda$$

$$E_t[(\phi_{t+1}(1+R_{W,t+1}))^\alpha]^{1/\alpha-1} E_t\left[\phi^\alpha_{t+1}(1+R_{W,t+1})^{\alpha}\right] = \lambda$$

$$E_t[(\phi_{t+1}(1+R_{W,t+1}))^\alpha]^{1/\alpha} = \lambda$$

Plug this back into the FOC (before we summed across assets) to get

$$E_t[(\phi_{t+1}(1+R_{W,t+1}))^\alpha]^{1/\alpha-1} E_t\left[\phi^\alpha_{t+1}(1+R_{W,t+1})^{\alpha-1}(1+R_{i,t+1})\right] = E_t[(\phi_{t+1}(1+R_{W,t+1}))^\alpha]^{1/\alpha}$$

$$E_t\left[\phi^\alpha_{t+1}(1+R_{W,t+1})^{\alpha-1}(1+R_{i,t+1})\right] = E_t[(\phi_{t+1}(1+R_{W,t+1}))^\alpha]$$

Plug in the expression of ϕ in terms of $\psi = C/W$

$$E_t\left[(1-\delta)^{\alpha/\rho}\left(\frac{C_{t+1}}{W_{t+1}}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha-1}(1+R_{i,t+1})\right] = E_t\left[(1-\delta)^{\alpha/\rho}\left(\frac{C_{t+1}}{W_{t+1}}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha}\right]$$

$$E_t\left[\left(\frac{C_{t+1}}{(W_t-C_t)(1+R_{W,t+1})}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha-1}(1+R_{i,t+1})\right]=E_t\left[\left(\frac{C_{t+1}}{(W_t-C_t)(1+R_{W,t+1})}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha}\right]$$

$$E_t\left[\left(\frac{C_{t+1}}{1+R_{W,t+1}}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha-1}(1+R_{i,t+1})\right]=E_t\left[\left(\frac{C_{t+1}}{1+R_{W,t+1}}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha}\right]$$

$$E_t\left[(C_{t+1})^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha/\rho-1}(1+R_{i,t+1})\right]=\underbrace{E_t\left[(C_{t+1})^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha/\rho}\right]}_{=\delta^{-\rho/\alpha}C_t^{\alpha\frac{\rho-1}{\rho}}\text{ from FOC for }C}$$

$$E_t\left[(C_{t+1})^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\alpha/\rho-1}(1+R_{i,t+1})\right]=\delta^{-\rho/\alpha}C_t^{\alpha\frac{\rho-1}{\rho}}$$

$$E_t\left[\delta^{\rho/\alpha}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\frac{\alpha}{\rho}-1}(1+R_{i,t+1})\right]=1$$

Hence, the SDF is

$$M_{t+1}=\delta^{\rho/\alpha}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha\frac{\rho-1}{\rho}}(1+R_{W,t+1})^{\frac{\alpha}{\rho}-1}$$

Epstein-Zin SDF Using notation from John's textbook we can rewrite the SDF as

$$M_{t+1}=\left(\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\right)^{\theta}\left(\frac{1}{1+R_{w,t+1}}\right)^{1-\theta} \tag{7}$$

and innovations in logs are

$$\tilde{m}_{t+1}=-\frac{\theta}{\psi}\tilde{c}_{t+1}-(1-\theta)\tilde{r}_{w,t+1} \tag{8}$$

Both innovations to consumption and returns to the wealth portfolio are priced. Notice that the two are combined with weights θ and $1-\theta$. When $\theta=1 \implies \gamma=\frac{1}{\psi}$ we have the baseline consumption CAPM SDF $\tilde{m}_{t+1}=-\gamma\tilde{c}_{t+1}$. When we have $\gamma=1$ and $\psi\neq 1 \implies \theta=0$ and we have $\tilde{m}_{t+1}=-\tilde{r}_{w,t+1}$ in the *spirit* of traditional CAPM.

Risk Premium and Risk Free Rate Now we're going to derive risk premium and the risk free rate assuming **joint lognormality and homoskedasticity of returns and consumption growth**. This is complicated by the fact that in order to derive the risk free rate we first need to price the wealth portfolio wince the SDF that determines the risk free rate depends on innovations to returns on the wealth portfolio. First, use the formula for the risk premium under joint lognormality in equation (**??**)

$$\begin{aligned}E_t r_{i,t+1}-r_{f,t+1}+\frac{\sigma_i^2}{2}&=-\sigma_{imt}\\&=-cov_t(m_{t+1},r_{i,t+1})\\&=-cov_t(\tilde{m}_{t+1},r_{i,t+1})\\&=-cov_t(-\theta\frac{\tilde{c}_{t+1}}{\psi}-(1-\theta)\tilde{r}_{w,t+1},r_{i,t+1})\\&=\frac{\theta}{\psi}\underbrace{\sigma_{ic}}_{\text{CCAPM}}+(1-\theta)\underbrace{\sigma_{iw}}_{\text{"CAPM"}}\end{aligned} \tag{9}$$

As the SDF the risk premium nests both Consumption and "Traditional" CAPM with weights θ and $1-\theta$.

Applying equation (9) to the wealth portfolio itself ($i=w$) we get

$$E_t r_{w,t+1}-r_{f,t+1}+\frac{\sigma_w^2}{2}=\frac{\theta}{\psi}\sigma_{wc}+(1-\theta)\sigma_w^2$$

$$E_t r_{w,t+1}=r_{f,t+1}-\frac{\sigma_w^2}{2}+\frac{\theta}{\psi}\sigma_{wc}+(1-\theta)\sigma_w^2$$

Standard pricing equation for joint lognormal sdf and returns (**??**) applied to the risk free rate implies

$$\begin{aligned} 0 &= E_t m_{t+1} + E_t r_{i,t+1} + \frac{\sigma_m^2}{2} + \frac{\sigma_i^2}{2} + \sigma_{im} \\ 0 &= E_t m_{t+1} + r_{f,t+1} + \frac{\sigma_m^2}{2} \\ r_{f,t+1} &= -E_t m_{t+1} - \frac{\sigma_m^2}{2} \end{aligned}$$

Plug in the log SDF (not in innovations form) to get

$$\begin{aligned} m_{t+1} &= \theta \log(\delta) - \frac{\theta}{\psi}\Delta c_{t+1} - (1-\theta) r_{w,t+1} \\ r_{f,t+1} &= -E_t \left[\theta \log(\delta) - \frac{\theta}{\psi}\Delta c_{t+1} - (1-\theta) r_{w,t+1}\right] - \frac{1}{2} Var_t \left(-\frac{\theta}{\psi}\Delta c_{t+1} - (1-\theta) r_{w,t+1}\right) \\ &= -\theta \log(\delta) + \frac{\theta}{\psi} E_t \Delta c_{t+1} + (1-\theta) E_t r_{w,t+1} - \frac{1}{2}\frac{\theta^2}{\psi^2}\sigma_c^2 + \frac{(1-\theta)^2}{2}\sigma_w^2 - \frac{\theta(1-\theta)}{\psi}\sigma_{wc} \\ &= -\theta \log(\delta) + \frac{\theta}{\psi} E_t \Delta c_{t+1} + (1-\theta) E_t \left[r_{f,t+1} - \frac{\sigma_w^2}{2} + \frac{\theta}{\psi}\sigma_{wc} + (1-\theta)\sigma_w^2\right] - \frac{1}{2}\frac{\theta^2}{\psi^2}\sigma_c^2 - \frac{(1-\theta)^2}{2}\sigma_w^2 - \frac{\theta(1-\theta)}{\psi}\sigma_{wc} \end{aligned}$$

Move risk free rate $(1-\theta) r_{f,t+1}$ on the other side and divide through by θ

$$\begin{aligned} r_{f,t+1} &= -\log(\delta) + \frac{1}{\psi} E_t \Delta c_{t+1} + \frac{(1-\theta)}{\theta}\left[-\frac{\sigma_w^2}{2} + \frac{\theta}{\psi}\sigma_{wc} + (1-\theta)\sigma_w^2\right] - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_c^2 - \frac{(1-\theta)^2}{2\theta}\sigma_w^2 - \frac{1-\theta}{\psi}\sigma_{wc} \\ &= -\log(\delta) + \frac{1}{\psi} E_t \Delta c_{t+1} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_c^2 + \frac{(1-\theta)}{\theta}\left[-\frac{\sigma_w^2}{2} + (1-\theta)\sigma_w^2\right] - \frac{(1-\theta)^2}{2\theta}\sigma_w^2 \end{aligned}$$

$$r_{f,t+1} = -\log(\delta) + \frac{1}{\psi} E_t \Delta c_{t+1} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_c^2 + \frac{\theta - 1}{2}\sigma_w^2 \tag{10}$$

Notice that for a given ψ there isn't a quadratic term in γ that precluded us from solving the equity premium puzzle by increasing γ. Therefore, high value of γ doesn't mess up with risk-free rate anymore once we separated risk aversion fro EIS.

3.6 Extended Consumption CAPM

In the expression for risk premium we have σ_{ic} and σ_{iw} – covariance of return on the asset with consumption growth and return on the wealth portfolio. However, through the budget constraint conusmption and return on wealth are linked with one another and, therefore, σ_{ic} and σ_{iw} are also linked. Once we account for this, we can derive a different expression for the SDF to get a different perspective on the implications of Epstein-Zin utility function.

First, use the Campbell-Shiller approximation in news form (equation **??**) for the wealth portoflio and note that its dividends are exactly consumption

$$\begin{aligned} \tilde{r}_{w,t+1} &= (E_{t+1} - E_t)\sum_{j=0}^{\infty} \rho^j \Delta d_{t+1+j} - (E_{t+1} - E_t)\sum_{j=1}^{\infty} \rho^j r_{w,t+1+j} \\ &= (E_{t+1} - E_t)\sum_{j=0}^{\infty} \rho^j \Delta c_{t+1+j} - (E_{t+1} - E_t)\sum_{j=1}^{\infty} \rho^j r_{w,t+1+j} \end{aligned}$$

Assume that all second moments are constants. Then from risk premium on the wealth portfolio in equation 9 we can write $r_{w,t+1}$ as

$$E_t r_{w,t+1} = const + r_{f,t+1} = const + \frac{1}{\psi} E_t \Delta c_{t+1}$$

$$\implies \tilde{r}_{w,t+1} = \frac{1}{\psi}(E_{t+1} - E_t)\Delta c_{t+1} \implies \tilde{r}_{w,t+1+j} = \frac{1}{\psi}(E_{t+1} - E_t)\Delta c_{t+1+j}$$

Plug this into the return approximation to get

$$
\begin{aligned}
\tilde{r}_{w,t+1} &= (E_{t+1} - E_t)\sum_{j=0}^{\infty}\rho^j \Delta c_{t+1+j} - (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j \frac{1}{\psi}\Delta c_{t+1+j} \\
&= (E_{t+1} - E_t)\Delta c_{t+1} + \left(1 - \frac{1}{\psi}\right)\underbrace{(E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j \Delta c_{t+1+j}}_{\tilde{g}_{t+1}} \\
&= \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}
\end{aligned} \tag{11}
$$

What is the intuition for this result? First, since consumption is the dividend of the wealth portfolio, consumption in period $t+1$ affects return on wealth portfolio one-for-one holding the path of future consumption fixed. Increase in future consumption growth has two offsetting effects: it increases dividends but also increases discount rates. The net effect is positive when $\psi > 1 \implies \frac{1}{\psi} < 1$. In this case, bad news about future consumption ($\tilde{g}_{t+1} < 0$) reduce the value of the wealth portfolio: **this is going to be important for the long-run risk model**. In a special case when $\psi = 1$, two effects offset each other.

Consumption-Wealth Ratio Under $\psi = 1$ consumption-wealth ratio is constant. To see this note that that wealth-consumption ratio is price-dividend ratio. Hence, we can use the the result from chapter 5 about the price dividend ratio (omitting the constant)

$$
\begin{aligned}
d_t - p_t &= -E_t\sum_{j=0}^{\infty}\rho^j \Delta d_{t+1+j} + E_t\sum_{j=0}^{\infty}\rho^j r_{t+1+j} \\
c_t - w_t &= -E_t\sum_{j=0}^{\infty}\rho^j \Delta c_{t+1+j} + E_t\sum_{j=0}^{\infty}\rho^j r_{w,t+1+j} \\
&= -E_t\sum_{j=0}^{\infty}\rho^j \Delta c_{t+1+j} + E_t\sum_{j=0}^{\infty}\rho^j \left(const + \frac{1}{\psi}E_t\Delta c_{t+1+j}\right) \\
w_t - c_t &= \left(1 - \frac{1}{\psi}\right)E_t\sum_{j=0}^{\infty}\rho^j \Delta c_{t+1+j}
\end{aligned} \tag{12}
$$

Hence, when $\psi = 1$ consumption to wealth ratio is constant.

Back to Risk Premium We can use this expression for return to substitute into the SDF innovation

$$
\begin{aligned}
\tilde{m}_{t+1} &= -\frac{\theta}{\psi}\tilde{c}_{t+1} - (1-\theta)\tilde{r}_{w,t+1} \\
&= -\frac{\theta}{\psi}\tilde{c}_{t+1} - (1-\theta)\left(\tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}\right) \\
&= -\left(1 + \theta\left(\frac{1}{\psi} - 1\right)\right)\tilde{c}_{t+1} - (1-\theta)\left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1} \\
&= -\gamma\tilde{c}_{t+1} - \left(\gamma - \frac{1}{\psi}\right)\tilde{g}_{t+1}
\end{aligned} \tag{13}
$$

Marginal utility moves both with contemporaneous movements in consumption and news about future consumption since it affects continuation utility. The second term reflect the aversion of the agent to *long-run risks*. Whenever, $\gamma > \frac{1}{\psi}$ good news about future consumption ($\tilde{g}_{t+1} > 0$ lower marginal utility today.

Using this sdf innovation we can write an alternative formula for the risk premium

$$
\begin{aligned}
E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} &= -\sigma_{im} \\
&= \gamma\sigma_{ic} + \left(\gamma - \frac{1}{\psi}\right)\sigma_{ig}
\end{aligned} \tag{14}
$$

Risk premium is determined by covariance with consumption and with news about future consumption.

3.7 Intertemporal CAPM

Strictly speaking intertemporal CAPM doesn't require Epstein-Zin Preferences.

Use the derived expression for $\tilde{r}_{w,t+1}$ to substitute

$$\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)(E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j \Delta c_{t+1+j} \implies (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j \Delta c_{t+1+j} = \frac{\psi}{\psi - 1}(\tilde{r}_{w,t+1} - \tilde{c}_{t+1})$$

to substitute out consumption growth from the return approximation

$$\begin{aligned}
\tilde{r}_{w,t+1} &= \tilde{c}_{t+1} + (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j \Delta c_{t+1+j} - (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{w,t+1+j} \\
&= \tilde{c}_{t+1} + \frac{\psi}{\psi - 1}(\tilde{r}_{w,t+1} - \tilde{c}_{t+1}) - (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{w,t+1+j} \\
&= \tilde{c}_{t+1}\left(1 - \frac{\psi}{\psi - 1}\right) + \frac{\psi}{\psi - 1}\tilde{r}_{w,t+1} - (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{w,t+1+j} \\
&= -\tilde{c}_{t+1}\frac{1}{\psi - 1} + \frac{\psi}{\psi - 1}\tilde{r}_{w,t+1} - (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{w,t+1+j} \\
\implies \tilde{c}_{t+1} &= r_{w,t+1} + (1 - \psi)\underbrace{(E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{w,t+1+j}}_{\tilde{h}_{t+1}}
\end{aligned}$$

$\tilde{c}_{t+1}$ reflects the direct effect of increase in wealth on consumption. The second term shows income net of substitution effect. 1 represents income effect: higher future expected returns increase lifetime income $\implies$ increase consumption. $\frac{1}{\psi}$ represents substitution effect: higher future return make it expensive to consume today, since can use the money for investments if future investment opportunities improve.

Now substitute this into the baseline SDF in equation (8) to get

$$\begin{aligned}
\tilde{m}_{t+1} &= -\frac{\theta}{\psi}\tilde{c}_{t+1} - (1 - \theta)\tilde{r}_{w,t+1} \\
&= -\frac{\theta}{\psi}\left(r_{w,t+1} + (1 - \psi)h_{t+1}\right) - (1 - \theta)\tilde{r}_{w,t+1} \\
&= -\gamma r_{w,t+1} - (\gamma - 1)\tilde{h}_{t+1}
\end{aligned} \tag{15}$$

marginal utility responds to both unexpected returns on the wealth portfolio as we've already seen but to also to changes in future investment opportuinities. When $\gamma > 1$ improvement in future investment opportunities ($\tilde{h}_{t+1} \uparrow$) this is good news and it decreases marginal utility.

Plugging this expression into the expression for the risk premium

$$\begin{aligned}
E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} &= -\sigma_{im} \\
&= \gamma\sigma_{iw} + (\gamma - 1)\sigma_{ih}
\end{aligned} \tag{16}$$

3.8 Main Equations of the Epstein-Zin Preferences

Below I outline the main equations from this section that highlight different logic

$$\begin{aligned} \text{Baseline SDF: } \tilde{m}_{t+1} &= -\gamma \tilde{c}_{t+1} - (\tilde{u}_{t+1} - \tilde{c}_{t+1}) \\ \text{Wealth Return SDF: } \tilde{m}_{t+1} &= -\theta \frac{\tilde{c}_{t+1}}{\psi} - (1-\theta)\tilde{r}_{w,t+1} \\ \text{Extended Consumption CAPM SDF: } \tilde{m}_{t+1} &= -\gamma \tilde{c}_{t+1} - \left(\gamma - \frac{1}{\psi}\right)\tilde{g}_{t+1} \\ \text{Intertemporal CAPM SDF: } \tilde{m}_{t+1} &= -\gamma \tilde{r}_{w,t+1} - (\gamma - 1)\tilde{h}_{t+1} \end{aligned} \tag{17}$$

4 Long-Run Risk Model

The main idea of the long-run risk models is to use **extended consumption CAPM** introduced earlier and augment it with persistent consumption growth process and stochastic volatility. First consider the main effects in a model with homoskedasticity. Expected risk premium on the wealth portfolio is

$$E_t r_{w,t+1} - r_{f,t+1} + \frac{\sigma_w^2}{2} = \gamma cov_t(r_{w,t+1}, \tilde{c}_{t+1}) + \left(\gamma - \frac{1}{\psi}\right) cov_t\left(r_{w,t+1}, \tilde{g}_{t+1}\right)$$

Use the result for the return on the wealth portfolio in equation (11) to get

$$E_t r_{w,t+1} - r_{f,t+1} + \frac{\sigma_w^2}{2} = \gamma cov_t(\tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}, \tilde{c}_{t+1}) + \left(\gamma - \frac{1}{\psi}\right) cov_t\left(\tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}, \tilde{g}_{t+1}\right)$$

Assume that innovations in current consumption are not correlated with news about future consumption

$$E_t r_{w,t+1} - r_{f,t+1} + \frac{\sigma_w^2}{2} = \gamma \sigma_c^2 + \left(\gamma - \frac{1}{\psi}\right)\left(1 - \frac{1}{\psi}\right)\sigma_g^2 \tag{18}$$

As was discussed the contribution of $\gamma\sigma_c^2$ is small in the data. The second term, however, can be relatively large if the volatility of news about future consumption is large. But in order to generate large and positive risk premium on the wealth portfolio we need the product that multiplies $\tilde{g}_{t+1}$ to be positive and there are two components

1. When $\gamma > \frac{1}{\psi}$ investors are averse to news about future consumption growth. In equation (17) when $\gamma > \frac{1}{\psi}$ bad news about future consumption increase marginal utility

2. When $1 > \frac{1}{\psi} \implies \psi > 1$ from equation (11) when there are bad news about future consumption ($\tilde{g}_{t+1} < 0$) return on the wealth portfolio is unexpectedly lower $\implies$ wealth decreases.

We need both of these components to generate risk premium since we need (1) investors to be averse to long run risk and (2) wealth portfolio to be exposed to long run risk.

Critiques of Long-Run Risks Models There are several problems with the simplest setting

1. Recall the expression for consumption to wealth ratio in equation (12)

$$w_t - c_t = \left(1 - \frac{1}{\psi}\right) E_t \sum_{j=0}^{\infty} \rho^j \Delta c_{t+1+j}$$

   It says that as long as $\psi > 1$ that is needed for long-run risks consumption-to-wealth ratio predicts future consumption growth. However, there is no evidence of it in the data.

4.1 Volatility and Risk Premium

Let's consider the effect of volatility on risk premium with Epstein-Zin preferences to understand why Bansal and Yaron version of the model needs changing volatility. For this part abstract from long term shocks to consumption and assume that log-consumption follows random walk with drift

$$c_{t+1} = c_t + g + \varepsilon_{t+1} \implies \Delta c_{t+1} + g \text{ where } var(\Delta c_{t+1}) = var(\tilde{c}_{t+1}) = var(\varepsilon_{t+1}) \equiv \sigma^2$$

Assumption of random walk implies that news about future consumption walk are always zero $\implies \tilde{g}_{t+1} = 0$. Use equation for return derived from CS return approximation (**??**)

$$\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1} = \tilde{c}_{t+1}$$

Hence, the volatility of the return on wealth portfolio equals to the volatility of consumption growth. Hence, from the expression for risk premium for the wealth portfolio (applying equation (9) to the wealth portfolio):

$$E_t r_{w,t+1} - r_{f,t+1} + \frac{\sigma_c^2}{2} = \frac{\theta}{\psi}\sigma_{wc} + (1-\theta)\sigma_c^2$$

$$E_t r_{w,t+1} = r_{f,t+1} - \frac{\sigma_c^2}{2} + \frac{\theta}{\psi}\sigma_c^2 + (1-\theta)\sigma_c^2$$

We can use the expression for the risk free rate (apply equation (10) for $\tilde{r}_{w,t+1} = \tilde{c}_{t+1}$):

$$r_{f,t+1} = -\log(\delta) + \frac{1}{\psi}E_t\Delta c_{t+1} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_c^2 + \frac{\theta-1}{2}\sigma_c^2$$

Plug this expression back into the expected return on wealth portfolio from above:

$$\begin{aligned} E_t r_{w,t+1} &= -\log(\delta) + \frac{1}{\psi}E_t\Delta c_{t+1} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_c^2 + \frac{\theta-1}{2}\sigma_c^2 - \frac{\sigma_c^2}{2} + \frac{\theta}{\psi}\sigma_c^2 + (1-\theta)\sigma_c^2 \\ &= -\log(\delta) + \frac{g}{\psi} - \frac{\sigma^2}{2}(1-\gamma)\left(1 - \frac{1}{\psi}\right) \end{aligned}$$

where I skipped some simplification steps. This is the expression for **expected return on the wealth portfolio when consumption growth is iid**. Now let's use the price to dividend CS approximation one more time (in the same way as in equatio (12)) for a portfolio with dividends that equal to consumption and with expected return given by the expression above

$$\begin{aligned} c_t - w_t &= -E_t\sum_{j=0}^{\infty}\rho^j\Delta c_{t+1+j} + E_t\sum_{j=0}^{\infty}\rho^j r_{w,t+1+j} \\ &= -\sum_{j=0}^{\infty}\rho^j g + \sum_{j=0}^{\infty}\rho^j\left[-\log(\delta) + \frac{g}{\psi} - \frac{\sigma^2}{2}(1-\gamma)\left(1-\frac{1}{\psi}\right)\right] \\ &= \frac{g}{1-\rho} - \frac{\log(\delta)}{1-\rho} + \frac{g}{\psi(1-\rho)} - \frac{1}{1-\rho}\times\frac{\sigma^2}{2}(1-\gamma)\left(1-\frac{1}{\psi}\right) \\ \implies w_t - c_t &\propto \frac{\sigma^2}{2}(1-\gamma)\left(1-\frac{1}{\psi}\right) \end{aligned}$$

Hence, consumption-to-wealth ratio decreases with volatility when $(1-\gamma)$ and $(1-1/\psi)$ have the opposite. What is the intuition for these results?

- When we fix g we fix geometric average return. Higher σ then means higher average arithmetic return. When $\gamma > 1 \implies 1-\gamma < 0$ the agent sees a deterioration of investment opportuinities. If $\psi > 1 \implies 1 - 1/\psi > 0$, then the agent has strong intertemporal substitution motives and increases his consumption in response to worse investment opportuinities.

Bottomline: for $\gamma > 1, \psi > 1$ increase in consumption volatility decreases consumption to wealth ratio. Under the empirical specification of Bansal and Yaron (2006) persistent volatility further amplifies the equity premium.

5 Habit Formation Models

5.1 Ratio Habit Model

This model is due to Abel (????). He changes utility to depend not on consumption but on ratio of consumption C_t to habit X_t so that

$$U_t = E_t \sum_{j=0}^{\infty} \delta^j \frac{1}{1-\gamma} \left(\left(\frac{C_{t+j}}{X_{t+j}} \right)^{1-\gamma} - 1 \right)$$

Habit is the aggregate consumption $X_t = (\overline{C}_{t-1})^{\kappa}$ but the agent is representative and doesn't internalize the effect of his decision on habit level. Marginal utility is

$$u'(C_t) = \frac{1}{X_t} \left(\frac{C_t}{X_t} \right)^{-\gamma}$$

Therefore, the SDF

$$\begin{aligned}
M_{t+1} &\equiv \delta \frac{u'(C_{t+1})}{u'(C_t)} \\
&= \delta \frac{X_t}{X_{t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{X_t}{X_{t+1}} \right)^{-\gamma} \\
&= \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{X_t}{X_{t+1}} \right)^{1-\gamma} \\
&= \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{C_t}{C_{t-1}} \right)^{\kappa(\gamma-1)} \\
m_{t+1} &= \log(\delta) + \gamma \Delta c_{t+1} - \kappa(\gamma - 1)\Delta c_t
\end{aligned}$$

Under joint lognormality we have

$$E_t r_{i,t+1} + E_t m_{t+1} + \frac{1}{2}\sigma_i^2 + \frac{1}{2}\sigma_m^2 + \sigma_{im} = 0$$

$$r_{f,t+1} + E_t m_{t+1} + \frac{1}{2}\sigma_m^2 = 0$$

$$r_{f,t+1} = -E_t m_{t+1} - \frac{1}{2}\sigma_m^2 = -\log(\delta) - \gamma E_t \Delta c_{t+1} - \frac{\gamma^2 \sigma_c^2}{2}$$

However, risk premium is the same $-\sigma_{im} = \gamma \sigma_{ic}$ since the habit level is known at time t.

5.2 Campbell-Cochrane (1999) Model

The primary point of this model is to explain the equity volaility puzzle: how to explain a large volatility of equity return with a low volatility of consumption growth. Campbell-Cochrane model uses an absolute deviation of consumption from the habit level which allows it to generate time-varying risk aversion even with homoskedastic consumption growth $\Delta c_{t+1} = g + e_{c,t+1}$. Preferences are

$$u(C_t) = \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma}$$

Define the surplus ratio $S_t = \frac{C_t - X_t}{X_t}$ – how high is consumption relative to consumption. Relative Risk Aversion with this utility function is

$$\begin{aligned}
-\frac{C_t u''(C_t)}{u'(C_t)} &= \gamma \frac{C_t (C_t - X_t)^{-\gamma-1}}{(C_t - X_t)^{-\gamma}} \\
&= \gamma \frac{C_t}{C_t - X_t} \\
&= \frac{\gamma}{S_t}
\end{aligned}$$

Hence, the *effective* risk aversion depends on the surplus ratio.

Surplus Ratio Adjustment Need to come up with the adjustment process for habit. They work with path for log surplus consumption $s_t = \log(S_t)$:

$$s_{t+1} = (1-\phi)\bar{s} + \phi s_t + \lambda(s_t) e_{c,t+1}$$

term $\lambda(s_t)$ specifies the *pass-through* of consumption to the habit. This term allows X_t to respond to shocks to C_t to ensure that X_t doesn't fall below C_t where utility is not defined. This term is also responsible for time-varying volatility of the SDF that amplifies risk premium movements. This equation implies (somehow non-trivially) that current habit is a function of all past consumptions

$$x_t = \alpha + (1-\phi)\sum_{j=0}^{\infty} \phi^j c_{t-j}$$

so that it responds slowly and lineraly to log consumption.

SDF for CC Model We can write the SDF in the surplus notation by noticing that

$$u'(C_t) = (C_t - X_t)^{-\gamma} = C_t^{-\gamma}\left(\frac{C_t - X_t}{C_t}\right)^{-\gamma} = C_t^{-\gamma} S_t^{-\gamma}$$

$$M_{t+1} = \delta\left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \implies \tilde{m}_{t+1} = -\gamma\tilde{s} - \gamma\tilde{c}_{t+1} = -\gamma\lambda(s_t)\varepsilon_{c,t+1} - \gamma\varepsilon_{c,t+1}$$

Risk Free Rate Risk free rate calculated in a usual way

$$\begin{aligned}
r_{f,t+1} &= -E_t m_{t+1} - \frac{1}{2} var_t(m_{t+1}) \\
&= -E_t\left[\log(\delta) - \gamma\Delta s_{t+1} - \gamma\Delta c_{t+1}\right] - \frac{1}{2} var_t(-\gamma\tilde{s} - \gamma\tilde{c}_{t+1}) \\
&= -\log(\delta) + \gamma E_t[s_{t+1} - s_t] + \gamma g - \frac{\gamma^2\sigma_c^2}{2}(1+\lambda(s_t))^2 \\
&= -\log(\delta) + \gamma\left[(1-\phi)\bar{s} - (1-\phi)s_t\right] + \gamma g - \frac{\gamma^2\sigma_c^2}{2}(1+\lambda(s_t))^2 \\
&= -\log(\delta) + \gamma g + \gamma(1-\phi)(\bar{s} - s_t) - \frac{\gamma^2\sigma_c^2}{2}(1+\lambda(s_t))^2
\end{aligned}$$

There are three main terms

1. $-\log(\delta) + \gamma g$: standard CCAPM terms
2. $\gamma(1-\phi)(\bar{s} - s_t)$: when s_t goes down times are bad *now* since consumption is closer to the habit. However, over time habit will just so that the agent will get used to bad times. Therefore, *now* wants to borrow against the future which drive up the interest rate
3. $-\frac{\gamma^2\sigma_c^2}{2}(1+\lambda(s_t))^2$: in CC parametrization to match observed data, $\lambda(s_t)$ declines in s_t. In a recession when s_t is low, volatility is large (seems plausible) and there is an elevated demand of precautionary savings that drive interest rate down

In CC parametrization the last two effects exatly offset each other implying constant riskless rate.

Some Other Results that the Model Delivers

- Sensitivity function $\lambda(s_t)$ is parametrized such that $\lambda(s_t) \to \infty$ as $s_t \to 0$. Need this so that habit is very sensitive to consumption so that consumption doesn't fall below habit. This implies that as $s_t \to 0$ the volatility of SDF $\to \infty$.
- Habit is predetermined in the steady state so that $dx/dc = 0$ at $s_t = \bar{s}$ and habit is predetermined near the steady state so that $d(dx/dc)/ds = 0$ at $s_t = \bar{s}$. We get that dx/dc is a U-shaped function of s_t and is tangent to 0 in $s_t = s$.

- Calibrate model parameters (such as ϕ) to arrive at the steady state surplus of surplus $\overline{S} = 0.057$ meaning that habit $= 0.94$ of consumption. The steady state risk aversion is $\gamma/\overline{S} = 2/0.057 \approx 30$. This mean that this model resembles a power utility model with high risk aversion and, hence, doesn't solve the equity premium puzzle.

- Additionally, since risk aversion moves with consumption, it allows to generate large predictable movements in stock prices of the sort discussed in chapter 5. Moreover, it introduces a countercyclical risk premium: after the consumption was lowered closer to the habit level, effective risk aversion increases driving the risk premium up.